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NORMAL EMBEDDINGS OF SL(2)/Γ

(PLONGEMENTS NORMAUX DE SL(2)/Γ)

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INTRODUCTION

In this work we will study the normal embeddings of SL(2)/Γ over an algebraically closed field of charcteristic zero, where Γ is a finite subgroup of SL(2). By an embedding of SL(2)/Γ we mean an irreducible algebraic variety with an action of SL(2) and which contains SL(2)/Γ as an open orbit. Thus in particular, it is a three-dimensional variety.

One reason to study embeddings in general is that it gives us a way to construct varieties. Usually, one constructs varieties either by finding a subvariety of a known variety, or one glues together several varieties. The difficulty with the second method is that one must make sure that one glues in a compatible way; the calculations for this can be quite complicated. If one is constructing an embedding, however, one can use the open orbit and the action of the group to simplify this calculation. That is, one can glue several embeddings together in such a way that they are compatible with the open orbit.

The study of SL(2)/Γ-embeddings yields many interesting examples of three-dimensional algebraic varieties. For example, one can find non-projective smooth complete algebraic varieties with an action of a finite group such that the quotient by this action is not algebraic (See [8]).

A theory was developed by Luna and Vust [9] for classifying normal embeddings of G/H where G is a reductive group with a factorial coordinate ring and H is an algebraic subgroup of G. The idea has similarities to the classification of torus embeddings (see for example [5]). For torus embeddings one uses the result that any normal torus embedding is covered by affine open stable subvarieties [12]. In their case Luna and Vust showed that any normal G/H-embedding is covered by open affine subvarieties stable by Borel subgroups of G. One uses this fact to collect data about the local rings of possible orbits of embeddings. Then one must check which sets of such local rings form the set of local rings of orbits of a variety. For the case G=SL(2) and H={e} the calculations are carried out in [9]. Each such embedding is represented by a diagram which contains data about the set of local rings of the orbits.

In Chapter I we describe how to extend the calculation to the case G=SL(2) and H=Γ, a finite subgroup. We show explicitly how the diagrams of the data looks for each finite subgroup.

The rest of this work deals with translating questions about geometric properties of an embedding into numerical conditions on its data. This will allow us to describe an embedding more fully from its diagram.

For example in Chapter II we calculate for certain cases which embeddings are smooth. This translates into certain conditions on the data of the diagram. We carry out completely the calculations for Γ={e} or {±e} and give partial results for the other cases.

Chapter III is completely independent of the previous two chapters. Here we consider a Borel subgroup B of SL(2), and let Γ be a finite subgroup of B. Then an embedding of B/Γ is a rational surface. We restrict to the case of smooth complete embeddings and then use the theory of smooth rational complete surfaces to classify these embeddings.

In the Chapter IV, first we describe a geometric method to obtain certain smooth complete embeddings of SL(2)/Γ from the embeddings found in Chapter III. Then we find the diagrams for these embeddings. In doing this we learn much about the geometry of these embeddings. In the process we learn also how to blow up certain smooth embeddings. In the last section we give a list of the "minimal" smooth SL(2)/Γ-embeddings for Γ={e} or {±e}. By minimal we mean that they are not obtained by blowing up another smooth embedding at a closed orbit.

We give now the precise definition for embedding that will be used throughout this work. Let G be a connected algebraic group and let H be an algebraic subgroup. Then an embedding of the homogeneous space G/H is a G-variety X with an equivariant open injective morphism $i: G/H \hookrightarrow X$. Two G/H-embeddings (X_1, i_1) and (X_2, i_2) are considered equivalent if there exists an equivariant isomorphism $\varphi: X_1 \xrightarrow{\sim} X_2$ such that $\varphi \circ i_1 = i_2$. (That is, an embedding is considered with a base point: the image of H/H by the equivariant injective morphism.) When there is no confusion, I denote the embedding (X,i) simply by X.

This study of $SL(2)/\Gamma$-embeddings partially follows a program proposed by D. Luna.

CHAPTER I : NORMAL EMBEDDINGS OF SL(2,k)/Γ

§ 1. Classifying the normal embeddings of SL(2,k)/Γ

Let k be an algebraically closed field of characteristic zero. Let G denote SL(2,k) and Γ a finite subgroup of G. In this section we will describe the method of [9] for the special case of classifying the normal embeddings of G/Γ. Then in the following sections we will carry out the calculations.

A normal embedding is characterized by the local rings of its orbits. So in order to find all the normal embeddings of G/Γ, first one must find the set of possible local rings of orbits. Then one must find which of such local rings can be combined to form a variety. We denote

$$\mathcal{L}_1^n(G/\Gamma) = \{\text{local rings of non-open orbits of normal } G/\Gamma\text{-embeddings}\}.$$

The first step is to describe $\mathcal{L}_1^n(G/\Gamma)$.

We fix some notation.

We denote by k[G] the ring of regular functions on G and by k(G) its quotient field. There is an action of G (resp. Γ) on k(G) induced by left (resp. right) translation. We call $k(G)^\Gamma$ the subfield of k(G) of invariants by right translation by Γ. We denote

$$\mathcal{V}(G/\Gamma) = \{\text{discrete normalized geometric valuations of } k(G)^\Gamma \text{ over k stable by G}\}$$

and $$\mathcal{V}_1(G/\Gamma) = \{v \in \mathcal{V}(G/\Gamma) \mid {}^G k_v \cong k\}$$

where k_v is the residue field of v and ${}^G k_v$ is the subfield of G-invariants. (We call a valuation "geometric" if its valuation ring is the localization of an algebra of finite type.)

Now fix B a Borel subgroup of G. We denote

$\mathcal{P}$ = {eigenvectors of B (by left translation) of k(G)},

$\mathcal{P}^{(\Gamma)}$ = $\{f \in \mathcal{P} \mid f$ is an eigenvector of Γ (by right translation)$\}$,

and ${}^{B}\mathcal{D}(G/\Gamma)$ = {irreducible divisors of G/Γ stable by B}.

Since B is of codimension one in G, ${}^{B}\mathcal{D}(G/\Gamma)$ is the set of B-orbits in G/Γ. For $\Gamma = \{e\}$, therefore, ${}^{B}\mathcal{D}(G/\{e\}) \cong B\backslash G \cong \mathbb{P}^1$ (We write simply $\mathbb{P}^1$ for the projective line $\mathbb{P}^1(k)$). So for a general Γ, we can identify ${}^{B}\mathcal{D}(G/\Gamma)$ with $\mathbb{P}^1/\Gamma$.

Let $\mathcal{D} \subset \mathbb{P}^1/\Gamma$ be a cofinite set. We set

$$A(\mathcal{D}) = \{f \in k(G)^{\Gamma} \mid f = gh \text{ with } g \in k[G],\ h \in \mathcal{P}^{(\Gamma)}, \text{ and } v_D(h) = 0\ \forall\ D \in \mathcal{D}\}$$

where v_D is the valuation of $k(G)^{\Gamma} = k(G/\Gamma)$ associated to the divisor D of G/Γ.

It is clear that $A(\mathcal{D}) = A(\tilde{\mathcal{D}})^{\Gamma}$ where $\tilde{\mathcal{D}}$ is the inverse image of $\mathcal{D}$ by the morphism $\mathbb{P}^1 \to \mathbb{P}^1/\Gamma$ and

$$A(\tilde{\mathcal{D}}) = \{f \in k(G) \mid f=gh \text{ with } g \in k[G],\ h \in \mathcal{P}, \text{ and } v_{\tilde{D}}(h) = 0\ \forall\ \tilde{D} \in \tilde{\mathcal{D}}\}.$$

Also since the group $\mathcal{P}$ is generated by $\mathcal{P} \cap k[G]$, we see that $A(\tilde{\mathcal{D}})$ is the localization of k[G] by the multiplicative set $\{h \in \mathcal{P} \cap k[G]\ v_{\tilde{D}}(h) = 0,\ \tilde{D} \in \tilde{\mathcal{D}}\}$; in other words, since k[G] is factorial, $A(\tilde{\mathcal{D}})$ is the ring of regular functions on $G - \bigcup_{\tilde{D} \in \tilde{\mathcal{D}}} \tilde{D}$; it follows that $A(\mathcal{D})$ is the ring of regular functions on $G/\Gamma - \bigcup_{D \in \mathcal{D}} D$.

Let $\mathcal{W} = \{w_1, \ldots, w_\alpha\} \subset \mathcal{V}(G/\Gamma)$. We denote

$$A(\mathcal{D},\mathcal{W}) = A(\mathcal{D}) \cap \mathcal{O}_{w_1} \cap \ldots \cap \mathcal{O}_{w_\alpha}$$

where $\mathcal{O}_w$ is the valuation ring of $w \in \mathcal{V}(G/\Gamma)$. Set $\widetilde{\mathcal{W}}$ equal to the set of valuations of $\mathcal{V}(G/\{e\})$ whose restrictions to $k(G/\Gamma)$ (up to normalization) are in $\mathcal{W}$. We will see in Lemma 1.2.1 that each valuation of $\mathcal{V}(G/\Gamma)$ can be extended to a valuation of $\mathcal{V}(G/\{e\})$. Using this fact, it is easy to see that $A(\mathcal{D},\mathcal{W}) = A(\widetilde{\mathcal{D}},\widetilde{\mathcal{W}})^\Gamma$. In [9], it is shown that $A(\widetilde{\mathcal{D}},\widetilde{\mathcal{W}})$ is an integrally closed sub-algebra of finite type of $k(G)$; therefore $A(\mathcal{D},\mathcal{W})$ is also an integrally closed subalgebra of finite type of $k(G/\Gamma)$.

If the fraction field of $A(\mathcal{D},\mathcal{W})$ is all of $k(G/\Gamma)$ and if $v \in \mathcal{V}_1(G/\Gamma)$ is positive on $A(\mathcal{D},\mathcal{W})$, then in [9] it is shown that the localization of $A(\mathcal{D},\mathcal{W})$ by the multiplicative set $S(\mathcal{D},\mathcal{W},v) = A(\mathcal{D},\mathcal{W}) \cap \mathcal{O}_v^*$ belongs to $\mathcal{L}_1^n(G/\Gamma)$. ($\mathcal{O}_v^*$ denotes the set of units in $\mathcal{O}_v$.) Also each element of $\mathcal{L}_1^n(G/\Gamma)$ can be obtained by this construction (geometrically this means that for any orbit of an embedding X, there is an affine B-stable open subvariety which intersects the orbit. So the list of elements of $\mathcal{L}_1^n(G/\Gamma)$ is given by the possible combinations of $\mathcal{D} \subset \mathbb{P}^1/\Gamma$, $\mathcal{W} \subset \mathcal{V}(G/\Gamma)$ and $v \in \mathcal{V}_1(G/\Gamma)$. Since one does not lose any information by restricting the valuations v_D for $D \in {}^B\mathcal{D}(G/\Gamma)$ and $w \in \mathcal{V}(G/\Gamma)$ to the subgroup $P^{(\Gamma)}$ of $k(G)$, we express the conditions for $(\mathcal{D},\mathcal{W},v)$ in the group of linear forms on $P^{(\Gamma)}/k^*$. For example, the condition : "the fraction field of $A(\mathcal{D},\mathcal{W})$ is $k(G/\Gamma)$" is equivalent to

(W) There exists $f \in P^{(\Gamma)} \cap A(\mathcal{D})$ such that $w(f) > 0$ for all $w \in \mathcal{W}$;

the condition "$\mathcal{O}_v$ contains $A(\mathcal{D},\mathcal{W})$ for $v \in \mathcal{V}_1(G/\Gamma)$" is equivalent to

(V) For all $f \in A(\mathcal{D},\mathcal{W}) \cap P^{(\Gamma)}$, we have $v(f) \geq 0$

(See [9].)

Now we must decide when two elements ℓ and ℓ' of $L_1^n(G/\Gamma)$ constructed as described above from $(\mathcal{D},\mathcal{W},v)$ and $(\mathcal{D}',\mathcal{W}',v')$ are the same. To do this, note that for $\ell \in L_1^n(G/\Gamma)$, $\mathcal{O}_\ell$ is a Krull ring, and its essential valuations consist of

(1) a finite subset $\mathcal{V}_\ell$ of $\mathcal{V}(G/\Gamma)$

and (2) a subset of the set of essential valuations of $k[G/\Gamma]$.

The set of essential valuations of $k[G/\Gamma]$ contains the set of v_D, $D \in {}^B\mathcal{D}(G/\Gamma)$; we denote ${}^B\mathcal{D}_\ell$ those v_D which are essential for $\mathcal{O}_\ell$.

Now we use the following important fact : Let $\mathcal{D} \subset {}^B\mathcal{D}(G/\Gamma)$ and $\mathcal{W} \subset \mathcal{V}(G/\Gamma)$; then there is at most one $\ell \in L_1^n(G/\Gamma)$ such that ${}^B\mathcal{D}_\ell = \mathcal{D}$ and $\mathcal{V}_\ell = \mathcal{W}$.

So given the triple $(\mathcal{D},\mathcal{W},v)$ satisfying (W) and (V), we want to find ${}^B\mathcal{D}_\ell$ and $\mathcal{V}_\ell$ where $\ell = \ell(\mathcal{D},\mathcal{W},v)$ is the element of $L_1^n(G/\Gamma)$ constructed from $(\mathcal{D},\mathcal{W},v)$. We have for $A(\mathcal{D},\mathcal{W})$:

(a) the v_D's for $D \in \mathcal{D}$ are essential for $A(\mathcal{D},\mathcal{W})$;

$(b)_1$ if $\mathcal{W} = \{w\}$ then w is essential for $A(\mathcal{D},\mathcal{W})$;

$(b)_2$ if card $\mathcal{W} \geq 2$, then all the elements of $\mathcal{W}$ are essential for $A(\mathcal{D},\mathcal{W})$ if and only if the following condition is satisfied :

(W') for all $w \in \mathcal{W}$, there exists $f_w \in P^{(\Gamma)} \cap A(\mathcal{D})$ such that

$$w(f_{w'}) \begin{cases} > 0 \text{ if } w \in \mathcal{W} - \{w'\} \\ = 0 \text{ if } w = w' \end{cases} .$$

Then the essential valuations of $\mathcal{O}_\ell$ are the essential valuations of $A(\mathcal{D},\mathcal{W})$ which are zero on the multiplicative set $S(\mathcal{D},\mathcal{W},v) = A(\mathcal{D},\mathcal{W}) \cap \mathcal{O}_v^*$. We denote

$$\mathcal{D}(\mathcal{W},v) = \{D \in \mathcal{D} \mid P^{(\Gamma)} \cap A(\mathcal{D},\mathcal{W}) \cap \mathcal{O}_v^* \subset \mathcal{O}_{v_D}^*\} ;$$

then ${}^{B}\mathcal{D}_{\ell} = \mathcal{D}(\mathcal{W},v)$. As for $\mathcal{V}_{\ell}$, the condition "every $w \in \mathcal{W}$ is essential for ℓ, that is $\mathcal{W} = \mathcal{V}_{\ell}$" is equivalent to $(\mathcal{D},\mathcal{W},v)$ satisfies (W') and

$$(V')\quad \mathcal{P}^{(\Gamma)} \cap A(\mathcal{D},\mathcal{W}) \cap \mathcal{O}_v^* \subset \mathcal{O}_w^* \quad \text{for all} \quad w \in \mathcal{W}.$$

Any $\ell \in \mathcal{L}_1^n(G/\Gamma)$ is constructed from a triple $(\mathcal{D},\mathcal{W},v)$ with $\mathcal{D}$ cofinite in ${}^{B}\mathcal{D}(G/\Gamma)$, $\mathcal{W}$ finite in $\mathcal{V}(G/\Gamma)$ and $v \in \mathcal{V}_1(G/\Gamma)$. One can always choose $\mathcal{W} = \mathcal{V}_{\ell}$. (One cannot always choose $\mathcal{D} = {}^{B}\mathcal{D}_{\ell}$. For example, sometimes ${}^{B}\mathcal{D}_{\ell}$ is finite.) So the technique for classifying $\mathcal{L}_1^n(G/\Gamma)$ is :

(i) classify $\mathcal{V}(G/\Gamma)$ and $\mathcal{V}_1(G/\Gamma)$;

(ii) find the triples $(\mathcal{D},\mathcal{W},v)$, $\mathcal{D}$ cofinite in ${}^{B}\mathcal{D}(G/\Gamma)$, $\mathcal{W}$ finite in $\mathcal{V}(G/\Gamma)$, and $v \in \mathcal{V}_1(G/\Gamma)$ satisfying (W), (V), (W') and (V') (then $\mathcal{W} = \mathcal{V}_{\ell(\mathcal{D},\mathcal{W},v)}$);

(iii) for each triple $(\mathcal{D},\mathcal{W},v)$ of (ii), calculate $\mathcal{D}(\mathcal{W},v)$ (then ${}^{B}\mathcal{D}_{\ell(\mathcal{D},\mathcal{W},v)} = \mathcal{D}(\mathcal{W},v)$).

Once we have described $\mathcal{L}_1^n(G/\Gamma)$, we must decide which combinations of localities in $\mathcal{L}_1^n(G/\Gamma)$ can occur for an embedding of G/Γ. Given an embedding X of G/Γ we denote

$$L(X) = \{\ell \in \mathcal{L}_1^n(G/\Gamma) \mid \ell \text{ is a locality in } X\}\ .$$

So the question is for which subsets L of $\mathcal{L}_1^n(G/\Gamma)$ does there exist an embedding X such that $L = L(X)$.

Now $\mathcal{L}_1^n(G/\Gamma)$ is a topological space with the topology of Zariski. For any embedding X of G/Γ, L(X) is open and noetherian. Also given an element $\ell \in \mathcal{L}_1^n(G/\Gamma)$ we define the <u>facette</u> of ℓ to be

$$\mathcal{F}_{\ell} = \{v \in \mathcal{V}_1(G/\Gamma) \mid \mathcal{O}_v \text{ dominates } \mathcal{O}_{\ell}\}.$$

For any subset $L \subset \mathcal{L}_1^n(G/\Gamma)$, we say L is separated if the facettes of the elements in L are disjoint. Certainly for any embedding X of G/Γ, L(X) is separated, since X is separated.

In fact, one can show given a subset $L \subset \mathcal{L}_1^n(G/\Gamma)$, there is an embedding X of G/Γ such that $L = L(X)$ if and only if L is open, noetherian, and separated [9].

Given $\ell = \ell(\mathcal{D},\mathcal{W},v)$ such that $(\mathcal{D},\mathcal{W},v)$ satisfies (W), (V), (W'), (V'), we will find $\mathcal{F}_\ell$. By construction, $v \in \mathcal{F}_\ell$. In fact, one can show that $v' \in \mathcal{F}_\ell$ if and only if $(\mathcal{D},\mathcal{W},v')$ satisfies (V) and (V') and $\mathcal{D}(\mathcal{W},v') = {}^B\mathcal{D}_\ell$.

Now we will describe a basis for the topology in $\mathcal{L}_1^n(G/\Gamma)$. Given $\ell = \ell(\mathcal{D},\mathcal{W},v)$, the set of localities of $A(\mathcal{D},\mathcal{W})$ in $\mathcal{L}_1^n(G/\Gamma)$ form an open set. Conversely, one can show that given any open neighborhood U of ℓ there exists a $\mathcal{D}' \subset {}^B\mathcal{D}(B/\Gamma)$ cofinite such that ℓ is a locality of $A(\mathcal{D}',\mathcal{V}_\ell)$ and the set of localities of $A(\mathcal{D}',\mathcal{V}_\ell)$ is contained in U; in other words, the sets of localities of $A(\mathcal{D}',\mathcal{V}_\ell)$ for $\ell \in \mathcal{L}_1^n(G/\Gamma)$ and such that $(\mathcal{D}',\mathcal{V}_\ell)$ satisfies (W) and (W') and $\mathcal{D} \supset {}^B\mathcal{D}_\ell$ form a basis of the topology in $\mathcal{L}_1^n(G/\Gamma)$. We can describe this basis in terms of $\mathcal{D},\mathcal{W}$ and v. Given $(\mathcal{D},\mathcal{W},v)$ which satisfies (W), (V) and (W'), denote

$$\mathcal{W}(\mathcal{D},v) = \{w \in \mathcal{W} \mid P^{(\Gamma)} \cap A(\mathcal{D},\mathcal{W}) \cap \mathcal{O}_v^* \subset \mathcal{O}_w^*\} .$$

Given $\ell \in \mathcal{L}_1^n(G/\Gamma)$ and $\mathcal{D} \in {}^B\mathcal{D}(G/\Gamma)$ such that $(\mathcal{D},\mathcal{V}_\ell)$ satisfies (W) and (W') and $\mathcal{D} \supset {}^B\mathcal{D}_\ell$, denote by $L(\mathcal{D},\ell)$ the set of $\ell' \in \mathcal{L}_1^n(G/\Gamma)$ with the following property : there exists $v' \in \mathcal{V}_1(G/\Gamma)$ such that $(\mathcal{D},\mathcal{V}_\ell,v')$ satisfies (V) and such that ${}^B\mathcal{D}_{\ell'} = \mathcal{D}(\mathcal{V}_\ell,v')$ and $\mathcal{V}_{\ell'} = \mathcal{V}_\ell(\mathcal{D},v')$. Then $L(\mathcal{D},\ell)$ is the set of localities of $A(\mathcal{D},\mathcal{V}_\ell)$ in $\mathcal{L}_1^n(G/\Gamma)$. So finally we have : the family of sets $L(\mathcal{D},\ell)$ where $(\mathcal{D},\mathcal{V}_\ell)$ satisfies (W) and (W') and $\mathcal{D} \supset {}^B\mathcal{D}_\ell$ forms a basis of the Zariski topology of $\mathcal{L}_1^n(G/\Gamma)$.

§ 2. Description of $V(G/\Gamma)$

In [9] there is a description of $V(G)$ and $V_1(G)$. We will deduce a similar description of $V(G/\Gamma)$ and $V_1(G/\Gamma)$.

Lemma 1.2.1. The restriction of a valuation in $V(G)$ to the field $k(G)^\Gamma$ induces a surjection $\Phi : V(G) \to V(G/\Gamma)$. (Note that it could be necessary to renormalize after the restriction.) Furthermore, $v \in V_1(G)$ if and only if $\Phi(v) \in V_1(G/\Gamma)$.

Proof.

First we must show that if $v \in V(G)$, then v', the restriction of v to $k(G)^\Gamma$, is an element of $V(G/\Gamma)$ (after renormalization). It is obviously a discrete valuation of $k(G)^\Gamma$ over k stable by G. Also, it is geometric because Γ is finite. (If $\mathcal{O}_v$ is a localization of an algebra A of finite type over k, then $\mathcal{O}_{v'}$ is a localization of A^Γ, which is of finite type.) So Φ is well defined.

To show that Φ is surjective, note that the extension $k(G)^\Gamma \subset k(G)$ is finite. So if $v' \in V(G/\Gamma)$, there are a finite number of valuations over k(G) which extend v'. Since G is connected, they must be stable by G. Let v be one of these extensions; we will show that v is geometric and therefore in $V(G)$. In general, a discrete valuation of a field K of transcendence degree n over k is geometric if and only if the transcendence degree of its residue field over k is n-1 [9]. In our case, we have that the extension $k(G)^\Gamma \subset k(G)$ is algebraic, and therefore also the extension of the residue fields $k_{v'} \subset k_v$ is algebraic. Since v' is geometric, so is v, and thus $v \in V(G)$.

Now we prove the last claim. Let $v' = \Phi(v)$. Then $k_{v'} \subset k_v$ is an algebraic extension. I claim that the extension ${}^G k_{v'} \subset {}^G k_v$ is also algebraic. For if $f \in {}^G k_v$, let P(x) be the minimal monic polynomial of f over $k_{v'}$. For any $s \in G$, we apply s to the coefficients of the polynomial P. The resulting polynomial is of the same degree, is monic, and has f as a root. Therefore it is P; so the coefficients of P are in ${}^{G}k_{v'}$; hence f is algebraic over ${}^{G}k_{v}$. Now since $k \subset {}^{G}k_{v'} \subset {}^{G}k_{v}$ and since k is algebraically closed, we have ${}^{G}k_{v'} \cong k$ if and only if ${}^{G}k_{v} \cong k$. This finishes the proof of the lemma.

□

Now we fix a Borel subgroup B of G. Recall that ${}^{B}\mathcal{D}(G/\Gamma)$ is the set of irreducible divisors of G/Γ stable by B, and ${}^{B}\mathcal{D}(G/\Gamma) \cong \mathbb{P}^1/\Gamma$. If $D \in {}^{B}\mathcal{D}(G/\Gamma)$, let $\tilde{D}$ denote the inverse image of D by the canonical morphism $G \to G/\Gamma$. For each $D \in {}^{B}\mathcal{D}(G/\Gamma)$, choose $g_D \in k[G]$ such that g_D generates the ideal in $k[G]$ of functions zero on $\tilde{D}$. Also recall that $\mathcal{P}^{(\Gamma)}$ is the set of eigenvectors of B (by left translation) and Γ (by right translation) of $k(G)$, and $\mathcal{P} = \mathcal{P}^{(\{e\})}$.

<u>Lemma 1.2.2.</u> $\mathcal{P}^{(\Gamma)} = \{c \prod_{D \in \mathbb{P}^1/\Gamma} (g_D)^{n_D} \mid c \in k^*, n_D \in \mathbb{Z},$ almost all n_D's $= 0\}$.

<u>Proof.</u>

We know the lemma is true for $\Gamma = \{e\}$; that is, $\mathcal{P} = \{c \prod_{D' \in \mathbb{P}^1} (g_{D'})^{n_{D'}} \mid c \in k^*,\ n_{D'} \in \mathbb{Z},$ almost all $n_{D'}$'s $= 0\}$. Also up to a scalar factor, $g_D = \prod_{D' \in \tilde{D}} g_{D'}$. With this information, the lemma is obvious.

□

Given $D \in {}^{B}\mathcal{D}(G/\Gamma)$, we denote

$a(D)$ = number of irreducible components of $\tilde{D}$;

$$m(D) = \frac{\text{card } \Gamma}{a(D)} ;$$

and $$f_D = g_D^{m(D)} .$$

Clearly $f_D \in k[G]^{\Gamma}$.

Corollary 1.2.3. Let $v_1, v_2 \in V(G/\Gamma)$; suppose that $v_1(f_D) = v_2(f_D)$ for all $D \in \mathbb{P}^1/\Gamma$; then $v_1 = v_2$.

Proof.

We know that if v_1 and v_2 coincide on $P^{(\Gamma)} \cap k(G)^\Gamma$, then they are equal (see [9], section 7.4).

Suppose $v_1(f_D) = v_2(f_D)$ for all $D \in \mathbb{P}^1/\Gamma$. Choose $\tilde{v}_1$ and $\tilde{v}_2$ extensions of v_1 and v_2 to the field $k(G)$. Then since f_D is a power of g_D, we have $\tilde{v}_1(g_D) = \tilde{v}_2(g_D)$ for all $D \in \mathbb{P}^1/\Gamma$; so by the lemma, v_1 and v_2 coincide on $P^{(\Gamma)}$. Therefore v_1 and v_2 coincide on $P^{(\Gamma)} \cap k(G)^\Gamma$, which implies that $v_1 = v_2$.

□

So we can describe an element of $v \in V(G/\Gamma)$ by the set of integers $\{v(f_D)\}_{D\in \mathbb{P}^1/\Gamma}$. Certainly not all these integers are positive : if they were, v would be positive on $k[G]^\Gamma$. Also, as we will see in the following proposition, $v(f_D)$ is constant for all $D \in \mathbb{P}^1/\Gamma$ except perhaps one D_o, for which $v(f_{D_o}) \geq v(f_D)$ for all $D \in \mathbb{P}^1/\Gamma$. Now we renormalize the elements of $V(G/\Gamma)$ such that if $v \in V(G/\Gamma)$, $v(f_D)$ is almost always -1 (so v is a valuation with values in $\mathbb{Q}$).

Proposition 1.2.4. (a) Given a $D \in \mathbb{P}^1/\Gamma$ and an $r \in (-1, \frac{2}{a(D)} - 1] \cap \mathbb{Q}$ there exists a unique valuation $v(D,r) \in V(G/\Gamma)$ defined by

$$v(D,r)(f_{D_o}) = \begin{cases} r & \text{if } D = D_o \\ -1 & \text{if } D \neq D_o \end{cases} ;$$

(b) $V_1(G/\Gamma) = \{v(D,r) \mid D \in \mathbb{P}^1/\Gamma, r \in (-1, \frac{2}{a(D)} - 1] \cap \mathbb{Q}\}$;

(c) $V(G/\Gamma) - V_1(G/\Gamma)$ consists of one element, $v(\ ,-1)$ such that $v(\ ,-1)(f_D) = -1$ for all $D \in \mathbb{P}^1/\Gamma$.

Proof.

We know the result is true for $\Gamma = \{e\}$ [9] . Suppose $v \in V_1(G/\Gamma)$. Then $v = \Phi(v(D',r'))$ for $D' \in \mathbb{P}^1$ and $r' \in (-1,1] \cap \mathbb{Q}$

by Lemma 1.2.1. Then $v(f_{D_o}) = v(D',r')(f_{D_o})$ for all $D_o \in \mathbb{P}^1/\Gamma$. Now

$$v(D',r')(f_{D_o}) = \begin{cases} m(D)(1-a(D_o)+r') & \text{if } D' \in \tilde{D}_o \\ -\operatorname{card}\Gamma & \text{if } D' \notin \tilde{D}_o . \end{cases}$$

We renormalize $v = (v(D',r'))$ to get

$$v(f_{D_o}) = \begin{cases} \frac{1+r'}{a(D)} - 1 & \text{for one special } D \in \mathbb{P}^1/\Gamma \\ -1 & \text{for all other elements of } \mathbb{P}^1/\Gamma . \end{cases}$$

Let $r = \frac{1+r'}{a(D)} - 1$. Then $v = v(D,r)$. This proves (a) and (b) (using Lemma 1.2.1 and Corollary 1.2.3).

Also $V(G/\Gamma) - V_1(G/\Gamma) = \{\Phi(v(\ ,-1))\} = \{v(\ ,-1)\}$. This finishes the proof of the Lemma.

□

For each $D \in \mathbb{P}^1/\Gamma$, we denote $b(D) = \frac{2}{a(D)} - 1$.

The finite subgroups of $SL(2,k)$ are well known. Each such group is conjugate to one of the following : C_n, the cyclic group of order n, $n \in \mathbb{N}^+$; $\tilde{D}_n$, the binary dihedral group of order $4n$, $n \geq 2$; $\tilde{\mathbb{T}}$, the binary tetrahedral group of order 24; $\tilde{\mathbb{O}}$, the binary octohedral group of order 48; or $\tilde{\mathbb{I}}$, the binary icosahedral group of order 120 (see for example [13]).

We will now describe $V(G/\Gamma)$ for each of these cases. For this we must study how Γ acts on ${}^{B}\mathcal{D}(G) \cong \mathbb{P}^1$. Given $(a:b) \in \mathbb{P}^1$ and $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma \subset SL(2,k)$,

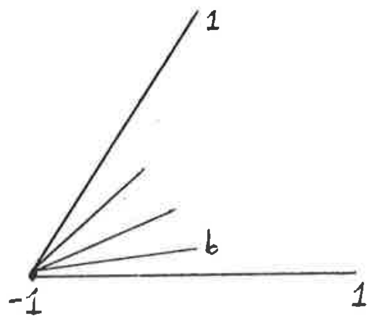
$$(a:b)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = (a\alpha + b\gamma : a\beta + b\delta) .$$

Since $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ acts trivially on $\mathbb{P}^1$, it is enough to study the action of $\overline{\Gamma}$, the image of Γ under the canonical morphism $SL(2,k) \to PGL(2,k)$. One can view $\mathbb{P}^1$ as the Riemann sphere and $\overline{\Gamma}$ as a group of rotations (if $k = \mathbb{C}$).

(1) Γ is conjugate to C_n.

(a) n odd :

There are two elements D_1 and D_2 of $\mathbb{P}^1$ fixed, and each other orbit is of order n. So for each $D \subset \mathbb{P}^1/\Gamma$ either $\tilde{D} = D_1$ or $\tilde{D} = D_2$, in which case $a(D) = 1$, or $\tilde{D}$ consists of n elements, in which case $a(D) = n$. So the diagram of $V(G/\Gamma)$ looks like

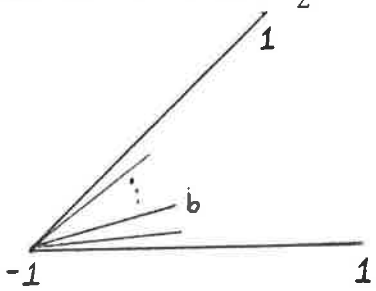


where $b = \frac{2}{n} - 1$.

Each branch represents a point of $\mathbb{P}^1/\Gamma$; the two long branches correspond to D_1 and D_2. The valuation $v(D,r)$ is represented by the point r on the branch corresponding to D.

(b) n even :

Here again two elements of $\mathbb{P}^1$ are fixed, but each other orbit is of order $\frac{n}{2}$. So the diagram of $V(G/\Gamma)$ looks like

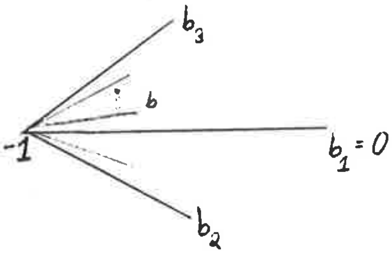


where $b = \frac{4}{n} - 1$.

For all the remaining cases, Γ contains the element $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. So the order of $\overline{\Gamma}$ is half the order of Γ. Also the group $\overline{\Gamma}$ is the group of rotations of the corresponding polyhedron. The vertices of the polyhedron form one orbit, the centers of the edges form another, and the centers of the faces form a third. Each other orbit is of the order of the group $\bar{\Gamma}$. So in general, there are three "special" elements of $\mathbb{P}^1/\Gamma$, D_1, D_2, D_3 with $a(D_1)$ = number of vertices of the polyhedron, $a(D_2)$ = number of edges, and $a(D_3)$ = number of faces. For the other elements D of $\mathbb{P}^1/\Gamma$, $a(D)$ = order of $\bar{\Gamma}$.

(2) Γ is conjugate to $\tilde{D}_n$.

Then $\bar{\Gamma}$ is of order 2n, and the dihedron has 2 vertices, n edges and n faces. So the diagram of $V(G/\Gamma)$ looks like



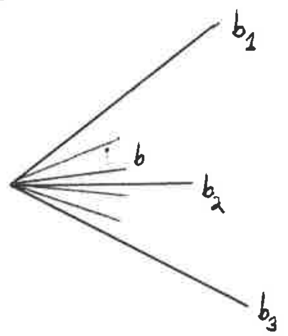
where $b_1 = 0$;

$b_2 = b_3 = \frac{2}{n} - 1$;

and $b = \frac{1}{n} - 1$.

(3) Γ is conjugate to $\tilde{\mathbb{T}}$.

Then $\bar{\Gamma}$ is of order 12, and the tetrahedron has 4 vertices, 6 edges, and 4 faces. So the diagram of $V(G/\Gamma)$ looks like



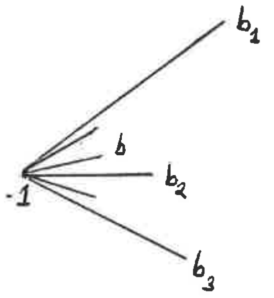
where $b_1 = b_3 = \frac{2}{4} - 1 = -\frac{1}{2}$;

$b_2 = \frac{2}{6} - 1 = -\frac{2}{3}$;

and $b = \frac{2}{12} - 1 = -\frac{5}{6}$.

(4) Γ is conjugate to $\tilde{\mathbb{O}}$.

Then the order of $\bar{\Gamma}$ is 24, and the octahedron has 6 vertices, 12 edges, and 8 faces. So the diagram of $V(G/\Gamma)$ looks like



where $b_1 = \frac{2}{6} - 1 = -\frac{2}{3}$;

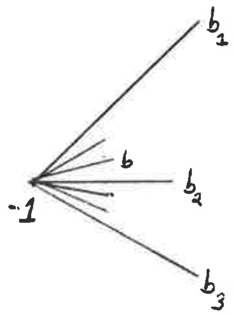
$b_2 = \frac{2}{12} - 1 = -\frac{5}{6}$;

$b_3 = \frac{2}{8} - 1 = -\frac{3}{4}$;

and $b = \frac{2}{24} - 1 = -\frac{11}{12}$.

(5) Γ is conjugate to $\tilde{I}$.

Then the order of $\bar{\Gamma}$ is 60, and the icosahedron has 12 vertices, 30 edges, and 20 faces. So the diagram of $V(G/\Gamma)$ looks like



where $b_1 = \frac{2}{12} - 1 = -\frac{5}{6}$;

$b_2 = \frac{2}{30} - 1 = -\frac{14}{15}$;

$b_3 = \frac{2}{20} - 1 = -\frac{9}{10}$;

and $b = \frac{2}{60} - 1 = -\frac{29}{30}$.

§ 3. Description of $L_1^n(G/\Gamma)$

Now we are ready to classify $L_1^n(G/\Gamma)$. The description one finds for $L_1^n(G/\Gamma)$ is very similar to the one given in [9] for $\Gamma = \{e\}$.

Recall that each element $\ell \in L_1^n(G/\Gamma)$ is the locality $\ell(\mathcal{D},\mathcal{W},v)$ for some $\mathcal{D} \subset {}^B\mathcal{D}(G/\Gamma)$ cofinite, $\mathcal{W} \subset V(G/\Gamma)$ finite and $v \in V_1(G/\Gamma)$ such that $(\mathcal{D},\mathcal{W},v)$ satisfies (W), (V), (W') and (V'). Furthermore, if $(\mathcal{D}',\mathcal{W}',v')$ satisfies these conditions also, then $\ell(\mathcal{D},\mathcal{W},v) = \ell(\mathcal{D}',\mathcal{W}',v')$ if and only if $\mathcal{W} = \mathcal{W}'$ and $\mathcal{D}(\mathcal{W},v) = \mathcal{D}'(\mathcal{W}',v')$.

First we state a proposition which describes the triples $(\mathcal{D},\mathcal{W},v)$ satisfying (W), (V), (W') and (V'). Then later we will find $\mathcal{D}(\mathcal{W},v)$ for each case.

Proposition 1.3.1. Let $\mathcal{D}$ be a cofinite subset of $\mathbb{P}^1/\Gamma$, $\mathcal{W} = \{w_1,\dots,w_\alpha\} \subset \mathcal{V}(G/\Gamma)$ with $w_j = v(D_j, r_j)$, $j = 1,\dots,\alpha$, and $v \in \mathcal{V}_1(G/\Gamma)$. Then $(\mathcal{D},\mathcal{W},v)$ satisfies (W), (V), (W') and (V') if and only if it is one of the following types :

Type A_α $(\alpha \geq 1)$

$$\mathcal{D} = \mathbb{P}^1/\Gamma - \{D_1,\dots,D_\alpha\} \text{ and } D_i \neq D_j \text{ if } i \neq j;\ -1 < r_j \leq b(D_j)$$

and $\sum_{j=1}^{\alpha} \frac{1}{1+r_j} < 1$; $v \in \bigsqcup_{D \in \mathcal{D}} v(D,]-1, b(D)]) \cup \bigsqcup_{j=1}^{\alpha} v(D_j,]-1, r_j[)$.

Type AB $(\alpha = 2)$

$D_1 \notin \mathcal{D}$ and $\mathcal{D} \neq \mathbb{P}^1/\Gamma - \{D_1\}$; $D_1 = D_2$ and

$-1 \leq r_1 < r_2 \leq b(D_1)$; $v \in v(D_1,]r_1, r_2[)$.

Type B_+ $(\alpha = 1)$

$D_1 \in \mathcal{D} \neq \mathbb{P}^1/\Gamma$; $-1 \leq r_1 < b(D_1)$; $v \in v(D_1,]r_1, b(D_1)])$.

Type B_- $(\alpha = 1)$

$\mathcal{D} = \mathbb{P}^1/\Gamma - \{D_1\}$; $0 < r_1 < b(D_1)$; $v \in v(D_1,]r_1, b(D_1)])$.

Type B_0 $(\alpha = 1)$

$\mathcal{D} = \mathbb{P}^1/\Gamma$; $0 < r_1 < b(D_1)$; $v \in v(D_1,]r_1, b(D_1)])$.

Type C

$\mathcal{W} = \{v\}$.

Proof.

The proof is practically identical to the proof given in [9] for the special case of $\Gamma = \{e\}$, so we will not give the details here.

To give one example of how to proceed, we will show : if $(\mathcal{D},\mathcal{W},v)$ is of type A_α, then the conditions given are verified. We check the four conditions :

(W) Case 1 : Either $\alpha \geq 3$ or $r_i < 1$, $i = 1,\ldots,\alpha$.
Then consider the function $F = (f_{D_1}\cdots f_{D_\alpha})^{-1}$. Clearly, $F \in A(\mathcal{D})$; also $w_i(F) = \alpha-1-r_i > 0$, $i = 1,\ldots,\alpha$; so the condition (W) is verified.

Case 2 : $\alpha = 2$ and $r_1 = 1$.
In this case $r_2 < 1$, since $(r_1+1)^{-1} + (r_2+1)^{-1} < 1$. Choose positive integers n_1 and n_2 such that $r_2 < n_1 n_2^{-1} < 1$; consider the function $F = f_{D_1}^{-n_1} f_{D_2}^{-n_2}$. Again $F \in A(\mathcal{D})$; also $w_1(F) = n_2 - r_1 n_1 = n_2 - n_1 > 0$ and $w_2(F) = n_1 - r_2 n_2 > 0$; so the condition (W) is verified.
If $\alpha = 1$, then $r_1 < 0$, since $(r_1+1)^{-1} < 1$; so this case is already covered in Case 1.

(W') Case 1 : There exists an $r_j \neq 1$.
For each $i \neq j$, let $f_{W_i} = (f_{D_i})^{-q_i}(f_{D_j})^{-p_i}$, where $r_i = \dfrac{p_i}{q_i}$, $p_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}^+$; then

$$w_i(f_{W_i}) = -q_i r_i + p_i = 0;$$

$$w_j(f_{W_i}) = -p_i r_j + q_i = q_i(1-r_i r_j) > 0 \text{ since } r_j \neq 1;$$

$$w_k(f_{W_i}) = q_i + p_i = q_i(1+r_i) > 0 \text{ for } k \neq i \text{ and } k \neq j.$$

Fix one $i_o \neq j$ between 1 and α, and let

$f_{W_j} = (f_{D_j})^{-q_j}(f_{D_{i_o}})^{-p_j}$; then one finds also

$$w_k(f_{W_j}) \begin{cases} = 0 & \text{if } k = j \\ > 0 & \text{if } k \neq j . \end{cases}$$

Clearly, f_{W_k} is in $A(\mathcal{D})$, $k = 1,\ldots,\alpha$; so the condition (W') is verified.

Case 2 : $\alpha \geq 3$ and $r_1 = \ldots = r_\alpha = 1$.

Let $f_{w_1} = (f_{D_1})^{-2}(f_{D_2})^{-1}(f_{D_3})^{-1}$, $f_{w_2} = (f_{D_2})^{-2}(f_{D_1})^{-1}(f_{D_3})^{-1}$, and for $i \geq 3$, let $f_{w_i} = (f_{D_i})^{-2}(f_{D_1})^{-1}(f_{D_2})^{-1}$; it is easy to verify that these f_{w_i}'s verify the property given in (W').

(V) and (V'). That these two conditions are verified is a direct consequence of the Lemma of section 9.3 [9].

□

Proposition 1.3.2. Suppose $(\mathcal{D},\mathcal{W},v)$ is of type A_α (resp. AB, B_+, B_-, B_o); then ${}^{B}\mathcal{D}_{\ell(\mathcal{D},\mathcal{W},v)} = \mathbb{P}^1/\Gamma - \{D_1,\ldots,D_\alpha\}$ (resp. ϕ, $\{D_1\}$, $\mathbb{P}^1/\Gamma - \{D_1\}$, $\mathbb{P}^1$).

Proof.

From section 1, we know it is enough to show that $\mathcal{D}(\mathcal{W},v)$ is as stated above for each type. This can be easily verified.

□

Let $\ell \in L_1^n(G/\Gamma)$. We define the type of ℓ to be the type of $(\mathcal{D},\mathcal{W},v)$ such that $\ell = \ell(\mathcal{D},\mathcal{W},v)$. One can read off the facette of ℓ from Proposition 1.3.1 : from section 1 we know given $\ell = \ell(\mathcal{D},\mathcal{W},v)$ of one of the types above, $v' \in F_\ell$ if and only if $\ell = \ell(\mathcal{D},\mathcal{W},v')$.

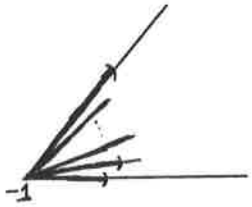
For each $\ell \in L_1^n(G/\Gamma)$ we draw a diagram which represents ℓ as follows :

(1) Darken the facette of ℓ in the diagram of $V(G/\Gamma)$;

(2) Since the facettes of elements of types B_+, B_- and B_o are the same, we distinguish the three cases by labelling the facette with a sign "+", "-" or "o".

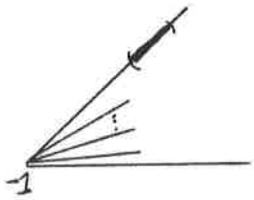
For example, if Γ is cyclic, the diagrams of the elements of $L_1^n(G/\Gamma)$ are as follows :

Type A_α

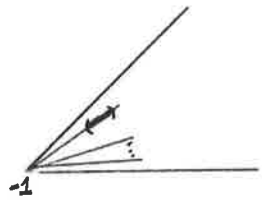


;

Type AB

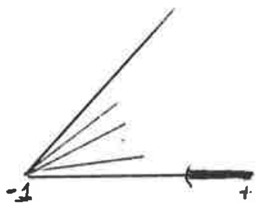


or

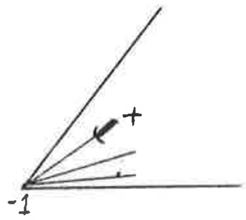


;

Type B_+

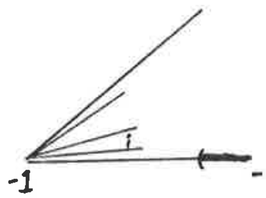


or



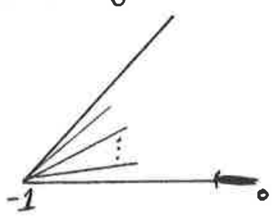
;

Type B_-



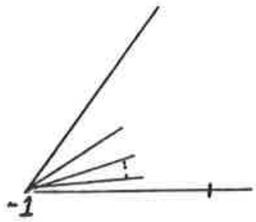
;

Type B_o

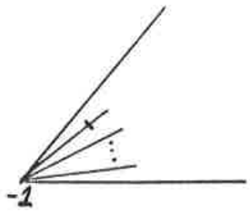


;

Type C



or



Note that for $\ell = \ell(\mathcal{D},\mathcal{W},v)$ of type B_- or B_0 with $\mathcal{W} = \{v(D,r)\}$, since $r > 0$, it is necessary that $b(D) = 1$; that is, the inverse image of D in $\mathbb{P}^1$ contains one element.

§ 4. The normal embeddings

We say that $L \subset \mathcal{L}_1^n(G/\Gamma)$ is a normal embedding if there exists a normal embedding X such that the set of localities of the non-open orbits of X, which we denote earlier as L(X), is equal to L. Using the remarks of section 1, we find which subsets of $\mathcal{L}_1^n(G/\Gamma)$ are normal embeddings. Again, the result is just like the result for the case $\Gamma = \{e\}$ given in [9].

First note that the type C localities in $\mathcal{L}_1^n(G/\Gamma)$ are exactly the valuations of $\mathcal{V}_1(G/\Gamma)$; so it makes sense to say that $\mathcal{V}_1(G/\Gamma)$ is contained in $\mathcal{L}_1^n(G/\Gamma)$. (This means that the elements of type C are the localities of orbits of codimension one.)

We denote

$$\mathcal{L}'(G/\Gamma) = \{\ell \in \mathcal{L}_1^n(G/\Gamma) \mid \ell \text{ is of type } B_+ \text{ and } \mathcal{V}_\ell = \{v(\ ,-1)\}\}.$$

Proposition 1.4.1. Let L be a subset of $L_1^n(G/\Gamma)$. Then L is a normal embedding if and only if it satisfies the following properties :

(i) if $\ell \in L$, then $V_\ell \cap V_1(G/\Gamma) \subset L$;

(ii) if $\ell \in L$ and $v(\ ,-1) \in V_\ell$, then L contains a subset cofinite in $L'(G/\Gamma)$;

(iii) $L - L'(G/\Gamma)$ is finite;

(iv) the facettes of the elements in L are disjoint.

To prove the proposition, one shows that "L is open in $L_1^n(G/\Gamma)$" is equivalent to (i) and (ii), "L is noetherian" is equivalent to (iii), and, as we already know from section 1, "L is separated" is equivalent to (iv). The proof is completely straightforward : the basis of $L_1^n(G/\Gamma)$ is given in section 1, and one uses Proposition 1.3.1 to find the open sets.

So instead of explicitly giving this proof, we will indicate the validity of the proposition by studying the geometrical structure of embeddings.

Let X be a normal embedding of G/Γ and $L = L(X)$. Then certainly L satisfies (i). Also, since $X-G/\Gamma$ has a finite number of components, X can have only a finite number of orbits of codimension one; in other words, L has only a finite number of type C localities; this fact together with property (i) implies property (iii). As for property (ii), suppose $\ell \in L$ with $v = v(\ ,-1) \in V_\ell$; then $X-G/\Gamma$ contains a component whose local ring is O_v. Let X_1 be the open subvariety of X obtained by removing all the other components of $X-G/\Gamma$. Then X_1 is an embedding of G/Γ, and $L(X_1) \subset L'(G/\Gamma)$. To see that $L(X_1)$ is cofinite in $L'(G/\Gamma)$, note that X_1 contains Spec $A(\mathcal{D},\{v\})$ for some $\mathcal{D}$ cofinite in $\mathbb{P}^1/\Gamma$; then it contains every element ℓ of $L_1^n(G/\Gamma)$ such that ${}^B\mathcal{D}_\ell \subset \mathcal{D}$; this set is cofinite. Another more constructive way to see the cofiniteness of $L(X_1)$ is to exhibit the embedding $Y(\Gamma)$ such that $L(Y(\Gamma)) = L'(G/\Gamma)$. Then $Y(\Gamma) - G/\Gamma$ is irreducible, and we will see that all its orbits are isomorphic to $\mathbb{P}^1$. Now $X_1 - G/\Gamma$ is a G-stable nonempty open subvariety of $Y(\Gamma)-G/\Gamma$, so $Y(\Gamma)-X_1$ is the union of a finite number of orbits.

One can construct $Y(\Gamma)$ as follows. First we construct $Y(\{e\})$. Let $M(2)$ denote the vector space of 2×2 matrices with coefficients in k. Let $Y(\{e\})$ be the closed subvariety of $\mathbb{P}(M(2) \oplus k)$ defined by the equation $\det(A) - t^2 = 0$ where A is in $M(2)$ and t is the coordinate on k. Now G acts on $Y(\{e\})$ by multiplication on $M(2)$, and there is an obvious equivariant inclusion of G into $Y(\{e\})$ defining an embedding. It is not difficult to see that $L(Y\{e\}) = L'(G/\Gamma)$ [9]. Also, $Y(\{e\})$ is a $G \times G$ variety where the second G acts by multiplication on the right of $M(2)$; therefore, for any subgroup Γ of G, Γ acts (by right multiplication) on $Y(\{e\})$. In fact it is not hard to show that for Γ a finite subgroup of G, the quotient $Y(\{e\})/\Gamma$ exists and is equal to $Y(\Gamma)$. One can show that $Y(\Gamma)-G/\Gamma$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1/\Gamma$, and each orbit is of the form $\mathbb{P}^1 \times \{a\}$, $a \in \mathbb{P}^1/\Gamma$.

To show the converse, first note the following :

(a) if $\ell \in L_1^n(G/\Gamma)$ and $v(\ ,-1) \notin V_\ell$, then there exists an embedding X such that $L(X) = \ell \cup V_\ell$;

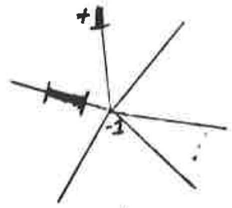
(b) if $\ell \in L_1^n(G/\Gamma)$ and $v(\ ,-1) \in V_\ell$, then there exists an embedding X such that $L(X) \subset \ell \cup V_\ell \cup L'(G/\Gamma)$.

For both (a) and (b), let X_o be any normal embedding such that $\ell \in L(X_o)$. By removing the irreducible components of $X_o - G/\Gamma$ with local rings not in V_ℓ, we can suppose that the only valuations in $L(X_o)$ are those in V_ℓ. In other words, suppose Y is the orbit of X_o with locality ℓ; then every orbit of dimension two of X_o contains Y in its closure. As for the orbits of dimension zero or one, they are always closed : any orbit of dimension one is isomorphic to $\mathbb{P}^1$.

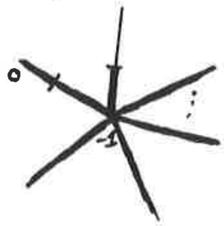
Since $L(X_o)$ satisfies property (iii), X_o contains a finite number of orbits of dimension zero or one not equal to Y whose localities are not in $\mathcal{L}'(G/\Gamma)$; set X equal to the open subvariety of X_o obtained by removing those orbits. It is clear that X satisfies (a) or (b) depending on whether $v(\ ,-1) \in \mathcal{V}_\ell$ or not.

Now suppose $L \subset \mathcal{L}_1^n(G/\Gamma)$ satisfies properties (i)-(iv). For each $\ell \in L$, construct a variety X_ℓ as follows : if $v(\ ,-1) \notin \mathcal{V}_\ell$, then X_ℓ is the variety defined in (a); if $v(\ ,-1) \in \mathcal{V}_\ell$, then X_ℓ is a variety which satisfies the condition in (b) and such that $\mathcal{L}(X_\ell) \subset L$ (this is possible by condition (ii)). Let Z be the subvariety of $Y(\Gamma)$ such that $L(Z) = \mathcal{L}'(G/\Gamma) \cap L$ (again this is possible by condition (ii)). Now glue these varieties together by identifying orbits with the same locality; this gives us a variety, X (it is separated by condition (iv) and noetherian by condition (iii)); clearly $L(X) = L$.

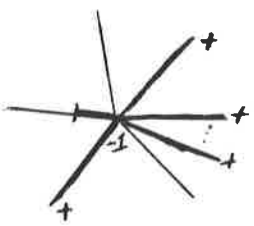
For each normal embedding L, we can draw a diagram by combining the diagrams of all the $\ell \in L$ on one copy of $\mathcal{V}(G/\Gamma)$. For example, if $\Gamma = \{e\}$ the following are examples of diagrams of embeddings :



(this embedding has 6 orbits);

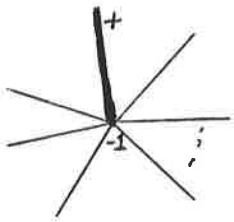


(this embedding has 5 orbits);

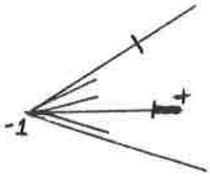


(this embedding has an infinite of orbits).

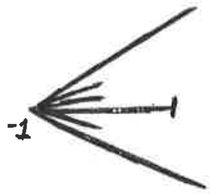
The following is not an embedding :



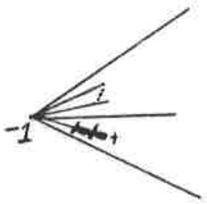
For $\Gamma =$ the following are examples of embeddings :



(this embedding has 4 orbits);



(this embedding has 3 orbits);



(this embedding has 5 orbits).

Note that in this case there are no orbits of type B_- or B_0 because $b(D) < 0$ for all D.

CHAPTER II : SMOOTH EMBEDDINGS OF $SL(2)/\Gamma$

In Chapter I, the normal embeddings of G/Γ are classified. In this chapter, we will show how to calculate which of these embeddings are smooth. If X is a normal embedding, then the set of singular points of X, Sing X, is stable by G. So each orbit of G in X is either entirely contained in Sing X or entirely contained in Reg X = X - Sing X. Certainly, any orbit of codimension one is contained in Reg X, since X is normal. Also, if $x \in X$ is a fixed point, Popov [10] showed that x is always singular. (The argument goes as follows. First note that x is contained in an affine open neighborhood, U, which is stable by G. Then the etale slice theorem of Luna [7] shows that if x were smooth, U must be a three-dimensional vector space, and the action of G must be linear. It can easily be shown that this is not the case, so x is not smooth.) It now remains to check to orbits of dimension one.

In the first section, we will state and prove some Lemmas. Then we will carry out the calculations completely for $\Gamma = \{e\}$ and $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Afterwards we will calculate some other examples for different Γ's.

§ 1. Methods to calculate the smooth embeddings

We denote

$$S(G/\Gamma) = \{\ell \in L_1^n(G/\Gamma) \mid \mathcal{O}_\ell \text{ is a regular local ring}\}.$$

We want to know which elements of $L_1^n(G/\Gamma)$ are in $S(G/\Gamma)$. Then an embedding X of G/Γ is smooth if and only if the localities of all its orbits are in $S(G/\Gamma)$.

Lemma 2.1.1.

(a) Suppose $\ell \in L_1^n(G/\Gamma)$ is of type B_o; then $\ell \notin S(G/\Gamma)$.

(b) Suppose $\ell \in L_1^n(G/\Gamma)$ is of type C; then $\ell \in S(G/\Gamma)$.

Proof.

The orbit associated to an element of type B_o is a fixed point. As remarked at the beginning of this chapter, fixed points are always singular.

The orbit associated to an element of type C is of dimension two, and it is therefore smooth.

□

All the other types of elements of $L_1^n(G/\Gamma)$ correspond to one-dimensional orbits. For any $\ell \in L_1^n(G/\Gamma)$, $\mathcal{O}_\ell$ is a local ring of a variety of dimension three. The following lemma will associate to each ℓ not of type B_o or C a local ring of a two-dimensional variety such that $\mathcal{O}_\ell$ is regular if and only if the associated ring is regular.

As in Chapter I, B is a fixed Borel subgroup of G. We denote by U the unipotent radical of B. For $\ell \in L_1^n(G/\Gamma)$ we construct a corresponding $A(\mathcal{D},\mathcal{W})$ as described in Chapter I. Since B, and therefore also U, act on $A(\mathcal{D},\mathcal{W})$, it makes sense to consider ${}^UA(\mathcal{D},\mathcal{W})$, the fixed elements of $A(\mathcal{D},\mathcal{W})$ under the action of U. In [9], it is shown that ${}^UA(\mathcal{D},\mathcal{W})$ is of finite type.

Lemma 2.1.2. Suppose $\ell \in L_1^n(G/\Gamma)$ and ℓ is the locality of a one-dimensional orbit. Then $A(\mathcal{D},\mathcal{W})$ is locally isomorphic to $k[U]\otimes_k {}^UA(\mathcal{D},\mathcal{W})$ in a neighborhood of $m_\ell \cap A(\mathcal{D},\mathcal{W})$ and $k[U]\otimes_k[m_\ell \cap {}^UA(\mathcal{D},\mathcal{W})]$.

Proof.

Let X be an embedding of G/Γ with orbit Y such that $\mathcal{O}_{X,Y} = \mathcal{O}_\ell$. Since Y is of dimension one, it is isomorphic to $\mathbb{P}^1$.

Pick a point $x \in Y$ such that the stabilizer of x is not B. In [3], it is shown that there exists an open B-stable affine neighborhood V of X such that $V \cong U \times U\backslash V$; that is, $k[V] \cong k[U] \otimes {}^{U}k[V]$. Let y be the image of $V \cap Y$ in the projection $V \to U\backslash V$.

Let $W = \operatorname{Spec} A(\mathcal{D},\mathcal{W})$. Then it makes sense to define a variety $U\backslash W = \operatorname{Spec} {}^{U}A(\mathcal{D},\mathcal{W})$, since ${}^{U}A(\mathcal{D};\mathcal{W})$ is of finite type. Let z_o be the image of $Y \cap W$ by the projection $W \to U\backslash W$.

To prove the Lemma, we show that $\mathcal{O}_{U\backslash V,y} = \mathcal{O}_{U\backslash W,z_o}$. By replacing V by $W \cap V$, we can suppose that $V \subset W$. Then since V is B-stable, $W-V$ is a union of closures of divisors in $\mathcal{D} - \mathcal{D}_\ell$. We set

$W-V = \bigcup_{i=1}^{n} \overline{D}_i$ where $D_i \in \mathcal{D} - \mathcal{D}_\ell$, $i = 1,\dots,n$. Since $D_i \in \mathcal{D}_\ell$ we know from Chapter I section 1 that there exists $g_i \in A(\mathcal{D},\mathcal{W}) \cap P^{(\Gamma)} \cap \mathcal{O}_\ell^*$ such that $v_D(g_i) > 0$, $i = 1,\dots,n$. Set $g = \prod_{i=1}^{n} g_i$. Clearly, $\mathcal{O}_{U\backslash W,z_o} \subset \mathcal{O}_{U\backslash V,y}$ since ${}^{U}k[W] \subset {}^{U}k[V]$. Now suppose $f \subset \mathcal{O}_{U\backslash V,y}$. Then $f = \frac{f_1}{f_2}$ with $f_1, f_2 \in {}^{U}k[V]$ and $f_2 \notin m_\ell$; so for some $N \in \mathbb{N}^+$ $f_1 g^N, f_2 g^N \in {}^{U}k[W]$ and also $f_2 g^N \notin m_\ell$ since neither f_2 nor g is in m_ℓ; so $\frac{f_1}{f_2} = \frac{f_1 g^N}{f_2 g^N} \in \mathcal{O}_{U\backslash W,z_o}$. This proves the Lemma.

□

From now on, we denote $Z(\mathcal{D},\mathcal{W}) = \operatorname{Spec} {}^{U}A(\mathcal{D},\mathcal{W})$, and z_o is the point of $Z(\mathcal{D},\mathcal{W})$ with local ring ${}^{U}\mathcal{O}_\ell$. This Lemma shows in particular that $\ell \in S(G/\Gamma)$ if and only if z_o is a smooth point of $Z(\mathcal{D},\mathcal{W})$; that is, $\ell \in S(G/\Gamma)$ if and only if ${}^{U}\mathcal{O}_\ell$ is a regular local ring.

One way to show that a certain ${}^{U}\mathcal{O}_\ell$ is regular is to find two elements that generate $m_\ell \cap {}^{U}A(\mathcal{D},\mathcal{W})$. The following Lemma deals with the generators of this ideal. As defined earlier, $P^{(\Gamma)}$ is the set of eigenvectors of B (by left translation) and of Γ (by right translation).

Lemma 2.1.3. Let $\ell \in L_1^n(G/\Gamma)$, and let $\mathcal{D}$ be a cofinite subset of ${}^B\mathcal{D}(G/\Gamma)$ and $\mathcal{W}$ a finite subset of $\mathcal{V}(G/\Gamma)$ such that $\mathcal{O}_\ell$ is a localization of $A(\mathcal{D},\mathcal{W})$. Then the ideal $m_\ell \cap {}^U A(\mathcal{D},\mathcal{W})$ is generated by $m_\ell \cap {}^U A(\mathcal{D},\mathcal{W}) \cap P^{(\Gamma)}$.

Proof.

Fix a torus $T \subset B$. Then $P^{(\Gamma)}$ contains the set of eigenvectors of T in ${}^U k(G)^\Gamma$. Since ${}^U A(\mathcal{D},\mathcal{W}) \subset A(\mathcal{D},\mathcal{W})$ is a rational T-module, it is generated by the eigenvectors of T. The ideal $m_\ell \cap {}^U A(\mathcal{D},\mathcal{W})$ is a T-stable ideal, so it is also generated by eigenvectors of T.

□

Another method we shall use is that in many cases ${}^U\mathcal{O}_\ell$ is not factorial and is therefore not regular (see e.g. [15]).

For any integral domain R, we say that $a \in R$ is <u>extremal in R</u> if it has no proper divisor in R : that is, if a = bc with $b,c \in R$, then either b or c is a unit. If S is a multiplicative submonoid of R, we say that a is <u>extremal in S</u> if it has no proper divisors in S : that is, if a = bc with $c \in R$, $b \in S$ and such that b is not a unit, then c is a unit in R (note that c need not be in S).

A noetherian domain R is factorial if and only if every extremal element of R is prime (we say $a \in R$ is prime if the ideal generated by a in R is prime).

The following Lemma will give us a method to prove that certain elements of ${}^U\mathcal{O}_\ell$ are extremal.

Lemma 2.1.4. Suppose X is a reduced irreducible affine variety on which a torus T acts rationally, and Y is a closed irreducible T-stable subvariety of X. Suppose also that $\mathcal{O}_{x,y}$ is integrally closed. Denote by P the set of eigenvectors of T in k(X). Let $f \in \mathcal{O}_{x,y} \cap P$; then f is extremal in $\mathcal{O}_{x,y} \cap P$ if and only if it is extremal in $\mathcal{O}_{x,y}$.

Proof.

Obviously, if f is extremal in $\mathcal{O}_{x,y}$, it is extremal in $\mathcal{O}_{x,y} \cap P$.

Now suppose f is extremal in $\mathcal{O}_{x,y} \cap P$. Let

$$f = g_1 \cdot g_2 \quad \text{with} \quad g_1, g_2 \in \mathcal{O}_{x,y}.$$

We will show that either g_1 or g_2 is a unit. We can suppose $g_1 \in k[X]$. The plan is to replace g_1 by an element of P. First we prove

Claim : If $t \in T$, then $t \cdot g_1 = ug_1$ where $u \in \mathcal{O}^*_{x,y}$.

Proof of claim : Since $\mathcal{O}_{x,y}$ is integrally closed, it is a Krull ring; we divide its essential valuations into two categories : those stable by T and those not stable by T. If w' is an essential valuation not stable by T, then $0 = w'(f) = w'(g_1) + w'(g_2)$; since g_1 and g_2 are in $\mathcal{O}_{x,y}$, we must have $w'(g_1) = 0$. Since $(t \cdot g_1)(t \cdot g_2) = t \cdot f = X(t)f$ where X is a character of T, we have similarly $w'(tg_1) = 0$. If w is a valuation stable by T, then $w(t \cdot g_1) = (t^{-1}w)(g_1) = w(g_1)$. So for all essential valuations of $\mathcal{O}_{x,y}$, the element $(t \cdot g_1)g_1^{-1} = u$ is a unit; therefore $u \in \mathcal{O}^*_{x,y}$, and the claim is proven.

The claim shows that the ideal $g_1\mathcal{O}_{x,y}$ is stable by T. Now consider the ideal $I = g_1\mathcal{O}_{x,y} \cap k[X]$ of $k[X]$. It is stable by T, so it is generated by elements of P. Suppose $h_1, \ldots, h_r \in I \cap P$ generate I. Then there exist $a_i \in \mathcal{O}_{x,y}$ such that $g_1 a_i = h_i$, $i = 1, \ldots, r$. Also $g_1 \in I$; so there exist $b_i \in k[X]$, $i = 1, \ldots, r$ such that $g_1 = \sum_{i=1}^{r} b_i h_i$. We substitute $g_1 a_i$ for h_i, and we find $g_1 = (\sum_{i=1}^{r} b_i a_i) g_1$; so $\sum_{i=1}^{r} b_i a_i = 1$. The a_i's and b_i's are all elements of $\mathcal{O}_{x,y}$. So for at least one i, $a_i \in \mathcal{O}^*_{x,y}$, because otherwise 1 would be in the maximal ideal of $\mathcal{O}_{x,y}$. Then $a_i^{-1} \in \mathcal{O}^*_{x,y}$ and $g_1 = a_i^{-1} h_i$, $h_i \in P \cap \mathcal{O}_{x,y}$.

Suppose g_1 is not a unit in $\mathcal{O}_{x,y}$; then h_i is also not a unit. Since $f = g_1 g_2 = h_1(a_i^{-1}g_2)$ is extremal in $\mathcal{O}_{x,y} \cap P$ and $h_i \in \mathcal{O}_{x,y} \cap P$ and is not a unit, then $a_i^{-1}g_2 \in \mathcal{O}^*_{x,y}$; since $a_i \in \mathcal{O}^*_{x,y}$, we have $g_2 \in \mathcal{O}^*_{x,y}$. This finishes the proof of the lemma. □

Now for a given finite subgroup Γ of G, we find $S(G/\Gamma)$ using the following facts :

(1) If ℓ is of type C, $\ell \in S(G/\Gamma)$.
(2) If ℓ is of type B_o, $\ell \notin S(G/\Gamma)$.
} (Lemma 2.1.1)

(3) If ℓ is of another type

(i) $\mathcal{O}_\ell$ is regular if and only if ${}^U\mathcal{O}_\ell$ is regular (Lemma 2.1.2);

(ii) if ${}^U\mathcal{O}_\ell$ is not factorial, then it is not regular;

and (iii) if $m_\ell \cap {}^UA(\mathcal{D}, \mathcal{W})$ is generated by two elements, then ${}^U\mathcal{O}_\ell$ is regular.

We use Lemma 2.1.3 to find generators of $m_\ell \cap {}^UA(\mathcal{D},\mathcal{W})$, and we apply Lemma 2.1.4 to the case $X = Z(\mathcal{D}, \mathcal{W})$, $Y = z_o$ to show that certain ${}^U\mathcal{O}_\ell$'s are not factorial.

There are two other methods of calculation which can be very useful for certain finite subgroups Γ. First we will state a lemma which relates part of the calculation of $S(G/\Gamma_2)$ to that of $S(G/\Gamma_1)$, where Γ_1 is a normal subgroup of Γ_2. Then we will state a lemma which, when Γ is cyclic, shows that often we can use the theory of torus embeddings.

Lemma 2.1.5. Suppose Γ_1 and Γ_2 are finite subgroups of G, and Γ_1 is normal in Γ_2 of index s. Let $\ell \in L_1^n(G/\Gamma_2)$ be such that there exist s distinct elements $\tilde{\ell}_1,\ldots,\tilde{\ell}_s \in L_1^n(G/\Gamma_1)$ with $\mathcal{O}_{\tilde{\ell}_i}$ a localization of the integral closure of $\mathcal{O}_\ell$ in $k(G/\Gamma_1)$ for $i = 1,\ldots,s$. Then $\ell \in S(G/\Gamma_2)$ if and only if $\ell_i \in S(G/\Gamma_1)$ for any i.

Proof.

Choose $(\mathcal{D},\mathcal{W})$ satisfying (W) and (W') such that $\mathcal{O}_\ell$ is a localization of $A = A(\mathcal{D},\mathcal{W})$. Let $\tilde{A}$ be the integral closure of A in $k(G/\Gamma_1)$. Since A is of finite type, so is $\tilde{A}$. Also $\mathcal{O}_{\tilde{\ell}_1},\ldots,\mathcal{O}_{\tilde{\ell}_s}$ are localizations of $\tilde{A}$ such that $\mathcal{O}_{\tilde{\ell}_i} \cap k(G/\Gamma_2) = \mathcal{O}_\ell$. Now the inclusion $A \subset \tilde{A}$ induces a finite ramified covering of s leaves of Spec $\tilde{A} \to$ Spec A. Now $\mathcal{O}_\ell$ is the localization of a subvariety Y of Spec A, and $\mathcal{O}_{\tilde{\ell}_1},\ldots,\mathcal{O}_{\tilde{\ell}_s}$ are localizations of subvarieties of Spec A which lie over Y. So this finite covering is unramified over Y, and the lemma is proven.

□

For the next lemma, let Γ be a cyclic group. Then fix D_1 and D_2, two distinct elements of ${}^B\mathcal{D}(G/\Gamma)$ such that $b(D_i) = 1$, $i = 1,2$ (so if Γ is of order 1 or 2, then D_1 and D_2 are any two divisors, and if the order of Γ is greater than 2, then D_1 and D_2 are the two divisors fixed by the action of Γ). Suppose $\ell \in L_1^n(G/\Gamma)$ is the locality of a one-dimensional orbit. We construct $Z(\mathcal{D},\mathcal{W})$ as described earlier. Let S be the torus with coordinate ring $U_A(\mathbb{P}^1/\Gamma - \{D_1,D_2\}) = k[f_{D_1}, f_{D_1}^{-1}, g_{D_1}\ g_{D_2}, (g_{D_1}g_{D_2})^{-1}]$.

Lemma 2.1.6. Suppose $\mathcal{D} \supset \mathbb{P}^1/\Gamma - \{D_1, D_2\}$ and $\mathcal{W} \subset \{w_1, w_2\}$ with $w_1 = v(D_1, r_1)$ and $w_2 = v(D_i, r_2)$, $i = 1$ or 2; then $Z(\mathcal{D}, \mathcal{W})$ is an embedding of the torus S. Furthermore, if there exists a $v \in V_1(G/\Gamma)$ such that $\mathcal{O}_v$ dominates $\mathcal{O}_\ell$ and $v = v(D_i, r)$, $r > -1$ and $i = 1$ or 2, then the point of $Z(\mathcal{D}, \mathcal{W})$ fixed by S is z_o. (For a reference of torus embeddings see [5].)

Proof.

It is clear that ${}^U A(\mathcal{D}, \mathcal{W}) \subset k[S] = {}^U A(\mathbb{P}^1/\Gamma - \{D_1, D_2\})$; also the quotient field of ${}^U A(\mathcal{D}, \mathcal{W})$ is the same as the quotient field of k[S]. To show that $Z(\mathcal{D}, \mathcal{W})$ is an S-embedding, it remains to show that ${}^U A(\mathcal{D}, \mathcal{W})$ is stable by the action of S. We know that ${}^U A(\mathcal{D}, \mathcal{W})$ is generated by ${}^U A(\mathcal{D}, \mathcal{W}) \cap P^{(\Gamma)}$. Suppose $f \in {}^U A(\mathcal{D}, \mathcal{W}) \cap P^{(\Gamma)}$; then f is of the form

$$f = c\, f_{D_1}^{n_1} (g_{D_1} g_{D_2})^{n_2} \prod_{i=3}^{m} (f_{D_1} + a_i f_{D_2})^{n_i}, \quad c, a_i \in k^* .$$

(To see this, note that for $D \neq D_i$ $i = 1, 2$, $\prod_{\gamma \in \Gamma} g_D \gamma$ is of the form $f_{D_1} + a\, f_{D_2}$, $a \in k^*$.)

Now if $s \in S$, then

$$s \cdot f = c'\, f_{D_1}^{n_1} (g_{D_1} g_{D_2})^{n_2} \prod_{i=3}^{m} (f_{D_1} + b_i f_{D_2})^{n_i}, \quad c, b_i \in k^* .$$

By choice of w_1 and w_2, $w_i(s \cdot f) = w_i(f)$ for $i = 1, 2$; also $v_{D_i}(f) = v_{D_i}(s \cdot f)$, $i = 1, 2$; so ${}^U A(\mathcal{D}, \mathcal{W})$ is stable, and $Z(\mathcal{D}, \mathcal{W})$ is a torus embedding.

Also, for $v = v(D_i, r)$ $i = 1$ or 2, $v(f) = v(s \cdot f)$. So the ideal $m_v \cap {}^U A(\mathcal{D}, \mathcal{W})$ is stable by the action of the torus. If $\mathcal{O}_v$ dominates $\mathcal{O}_\ell$, this is the maximal ideal of the point z_o.

□

One has only to check the list in Chapter I of the types of elements of $L_1^n(G/\Gamma)$ to see when Lemma 2.1.6 is applicable. For types AB, B_+, B_-, A_1 and A_2 when $\mathcal{W} \subset \{v(D_1,r_1),\ v(D_i,r_2)\}$, $i = 1$ or 2, one can choose $\mathcal{D}$ such that $Z(\mathcal{D},\mathcal{W})$ is a torus embedding. Also, using the second part of Lemma 2.1.6, one can show that the fixed point of $Z(\mathcal{D},W)$ is z_o.

§ 2. Case of $\Gamma = \{e\}$.

For this case, we will calculate $S(G) = S(G/\{e\})$. Remember that ${}^B\mathcal{D}(G) \cong \mathbb{P}^1$, and $b(D) = 1$ for all $D \in {}^B\mathcal{D}(G)$. That is, in the diagram of $\mathcal{V}(G)$, all of the rays have the same length.

Proposition 2.2.1. Let $\ell \in L_1^n(G)$; then $\ell \in S(G)$ if and only if it is one of the following types :

1a) Type A_1 with $r_1 = -\frac{1}{q}$, $q \in \mathbb{N}^+$;

1b) Type A_2 with $r_1 = r_2 = 0$
or with $r_i = 1$ and $r_j = \frac{q-1}{q}$, $q \in \mathbb{N}^+$;

2) Type AB with $r_i = \frac{p_i}{q_i}$ and $|r_1 - r_2| = \frac{1}{q_1 q_2}$;

3) Type B_+ with $r_1 = 0$ or -1 ;

4) Type B_- with $r_1 = \frac{1}{q}$, $q \in \mathbb{N}^+$;

5) Type C.

(We make the conventon that if we write $r = \frac{p}{q}$, we choose $q > 0$ and $p,q \in \mathbb{Z}$ and relatively prime.)

To prove this proposition, we will use the lemmas of the previous section.

Proof.

First of all, if ℓ is of type C, then $\ell \in S(G)$ by Lemma 2.1.1(b). If ℓ is of type B_o, then $\ell \notin S(G)$ by Lemma 2.1.1(a).

In all the other cases, ℓ is the locality of an orbit of dimension one. We will divide the proof for these cases into two parts. First we will consider the types where we can apply Lemma 2.1.6 and use the theory of torus embeddings. Then later we will treat the other types.

Case 1 : ℓ is of type AB, B_+, B_-, A_1 or A_2.

One has only to check the list of $L_1^n(G)$ given in Chapter I to see that for the types listed above, the Lemma 2.1.6 applies. That is, one can choose $\mathcal{D} \subset \mathbb{P}^1 - \{D_1, D_2\}$ with $D_1 \neq D_2$ such that $\mathcal{W} \subset \{w_1, w_2\}$ with $w_1 = v(D_1, r_1)$ and $w_2 = v(D_i, r_2)$ for $i = 1$ or 2. Recall that we denote $Z(\mathcal{D}, \mathcal{W}) = \operatorname{Spec} {}^U A(\mathcal{D}, \mathcal{W})$. By Lemmas 2.1.2 and 2.1.6, $\ell \in S(G)$ if and only if $Z(\mathcal{D}, \mathcal{W})$ is a smooth torus embedding.

Now we will review some facts about torus embeddings which will allow us to calculate for which ℓ the corresponding $Z(\mathcal{D}, \mathcal{W})$ is smooth.

We call the two-dimensional torus for which $Z(\mathcal{D}, \mathcal{W})$ is an embedding S. We denote

$$X(S) = \{\text{characters of } S\} \cong \mathbb{Z}^2$$

and $$X_*(S) = \{\text{one-parameter subgroups of } S\} \cong \mathbb{Z}^2.$$

Then $X(S)$ and $X_*(S)$ are dual. The vector space of linear functionals on $X(S)$ is identified with $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. Let σ be the sector of $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ of linear functionals on $X(S)$ which are positive on $X(S) \cap {}^U A(\mathcal{D}, W)$. Then σ is generated over $\mathbb{R}^+$ by two primitive vectors, $e_1, e_2 \in X_*(S)$.

The theory tells us that the following conditions are equivalent :

(i) $Z(\mathcal{D},\mathcal{W})$ is smooth (and in fact $\cong A_k^2$);

(ii) $Z(\mathcal{D},\mathcal{W})$ is smooth at the fixed point;

(iii) $\{e_1,e_2\}$ forms a basis of $X_*(S)$ over $\mathbb{Z}$;

(iv) $\det(e_1\ e_2) = \pm 1$.

(See [5].)

The first step is to calculate the characters of the torus S. The ring of regular functions of S is given by $^U A(\mathbb{P}^1 - \{D_1,D_2\}) = k[f_{D_1}, f_{D_1}^{-1}, f_{D_2}, f_{D_2}^{-1}]$. So $X(S) = \{f_{D_1}^{m_1} f_{D_2}^{m_2} | m_1,m_2 \in \mathbb{Z}\}$.

Now for each type we must choose an appropriate $\mathcal{D}$; then we find e_1 and e_2 and calculate the determinant. Then by the above remarks, we will know when $\ell \in \mathcal{S}(G)$.

<u>Type AB</u>. Let $w_1 = v(D_1,r_1)$ and $w_2 = v(D_1,r_2)$ with $-1 \leq r_1 < r_2 \leq 1$, $r_i = \frac{p_i}{q_i}$, $i = 1,2$. We must choose $\mathcal{D} \subsetneq \mathbb{P}^1 - \{D_1\}$. So we choose $\mathcal{D} = \mathbb{P}^1 - \{D_1,D_2\}$ for an arbitrary $D_2 \in \mathbb{P}^1$. Now if f is a character of S, then $f \in {^U A}(\mathcal{D})$, so $f \in {^U A}(\mathcal{D},\mathcal{W})$ if and only if $w_i(f) \geq 0$, $i = 1,2$. So $X(S) \cap {^U A}(\mathcal{D},\mathcal{W}) = \{f_{D_1}^{m_1} f_{D_2}^{m_2} \mid m_1 r_i - m_2 \geq 0 \quad i = 1,2\}$. Then if σ is the sector of linear functionals positive on $X(S) \cap {^U A}(\mathcal{D},\mathcal{W})$, then σ is generated over $\mathbb{R}^+$ by $e_1 = (p_1,-q_1)$ and $e_2 = (p_2,-q_2)$. Then $\det(e_1\ \ e_2) = \det\begin{pmatrix} p_1 & p_2 \\ -q_1 & -q_2 \end{pmatrix} = -p_1q_2 + p_2q_1$. So if ℓ is of type AB, then $\ell \in \mathcal{S}(G)$ if and only if $|p_2q_1 - p_1q_2| = 1$, or in other words, $r_2 - r_1 = (q_1q_2)^{-1}$.

<u>Type B_+</u>. Let $w_1 = v(D_1, r_1)$, $-1 \leq r_1 = \frac{p_1}{q_1} < 1$. We must choose $\mathcal{D} \neq \mathbb{P}^1$ with $D_1 \in \mathcal{D}$. So let $\mathcal{D} = \mathbb{P}^1 - \{D_2\}$ for an arbitrary D_2. A character $f_{D_1}^{m_1} f_{D_2}^{m_2} \in {}^U A(\mathcal{D})$ if and only if $m_1 \geq 0$, and it belongs to ${}^U A(\mathcal{D}, \mathcal{W})$ if and only if $m_1 \geq 0$ and $w_1(f_{D_1}^{m_1} f_{D_2}^{m_2}) = m_1 r_1 - m_2 \geq 0$. So $X(S) \cap {}^U A(\mathcal{D}, \mathcal{W}) = \{ f_{D_1}^{m_1} f_{D_2}^{m_2} \mid m_1 \geq 0$ and $m_1 r_1 - m_2 \geq 0\}$. If σ is the sector of linear functionals positive on $X(S) \cap {}^U A(\mathcal{D}, \mathcal{W})$, then σ is generated by $e_1 = (1,0)$ and $e_2 = (p_1, -q_1)$. Then $\det(e_1 \;\; e_2) = \det\begin{pmatrix} 1 & p_1 \\ 0 & -q_1 \end{pmatrix} = -q_1$. Now $q_1 = 1$ if and only if $r_1 = 0$ or -1. So if ℓ is of type B_+, then $\ell \in S(G)$ if and only if $r_1 = 0$ or -1.

<u>Type B_-</u>. Let $w_1 = v(D_1, r_1)$, $0 < r_1 = \frac{p_1}{q_1} < 1$. We must choose $\mathcal{D} = \mathbb{P}^1 - \{D_1\}$. Pick an arbitrary $D_2 \in \mathcal{D}$. A character $f_{D_1}^{m_1} f_{D_2}^{m_2} \in {}^U A(\mathcal{D}, W)$ if and only if it belongs to $A(\mathcal{D})$ and $w_1(f_{D_1}^{m_1} f_{D_2}^{m_2}) = m_1 r_1 - m_2 \geq 0$. So $X(S) \cap {}^U A(\mathcal{D}, \mathcal{W}) = \{ f_{D_1}^{m_1} f_{D_2}^{m_2} \mid m_2 \geq 0$ and $m_1 r_1 - m_2 \geq 0\}$. In this case, σ is generated by $e_1 = (0,1)$ and $e_2 = (p_1, -q_1)$. Then $\det(e_1 \;\; e_2) = \det\begin{pmatrix} 0 & p_1 \\ 1 & -q_1 \end{pmatrix} = -p_1$. Since $r_1 > 0$, we always have $p_1 > 0$. So the locality of the orbit is smooth if and only if $p_1 = 1$, in other words, $r_1 = q_1^{-1}$. So if ℓ is of type B_-, then $\ell \in S(G)$ if and only if $r_1 = q_1^{-1}$, $q_1 \in \mathbb{N}^+$.

<u>Type A_1</u>. Let $w_1 = v(D_1, r_1)$ with $-1 < r_1 = \frac{p_1}{q_1} < 0$. We must choose $\mathcal{D} = \mathbb{P}^1 - \{D_1\}$. As in type B_-, choose any $D_2 \in \mathcal{D}$. Again $X(S) \cap {}^U A(\mathcal{D}, W) = \{ f_{D_1}^{m_1} f_{D_2}^{m_2} \mid m_2 \geq 0$ and $m_1 r_1 - m_2 \geq 0\}$. So again we find $\det(e_1 \;\; e_2) = -p_1$, but this time p_1 is negative.

Now $p_1 = -1$ signifies that $r_1 = -q_1^{-1}$. So if ℓ is of type A_1, then $\ell \in S(G)$ if and only if $r_1 = -q_1^{-1}$, $q_1 \in \mathbb{N}^+$.

Type A_2. Let $w_1 = v(D_1,r_1)$, $w_2 = v(D_2,r_2)$ with $-1 < r_i = \frac{p_i}{q_i} \leq 1$, $i = 1,2$, such that either r_1 or r_2 is not equal to 1. We must choose $\mathcal{D} = \mathbb{P}^1 - \{D_1,D_2\}$. In this case, $X(S) \cap {}^U A(\mathcal{D},\mathcal{W}) = \{f_{D_1}^{m_1} f_{D_2}^{m_2} \mid r_1m_1 - m_2 \geq 0 \text{ and } -m_1 + r_2m_2 \geq 0\}$. So σ is generated be $e_1 = (p_1,-q_1)$ and $e_2 = (-q_2,p_2)$; then $\det(e_1\ e_2) = \det\begin{pmatrix} p_1 & -q_2 \\ -q_1 & p_2 \end{pmatrix} = p_1p_2 - q_1q_2 < 0$. We check the conditions for $q_1q_2 - p_1p_2$ to be equal to one. Since $r_i > -1$, we have $p_i + q_i > 0$. If $q_1q_2 - p_1p_2 = 1$, we have $1 = q_1q_2 - p_1p_2 > -p_1(q_2+p_2)$, so $p_1 \geq 0$. Now $1 = q_1q_2 - p_1p_2 = (q_1-p_1)q_2 + (q_2-p_2)p_1$ if and only if

(i) $q_1 = p_1 = 1$ and $q_2 - p_2 = 1$

or (ii) $q_2 = p_2 = 1$ and $q_1 - p_1 = 1$

or (iii) $p_1 = 0$ and $q_2 = 1$.

This gives exactly what was stated in the proposition, and we have finished the proof for these types.

Now we consider the remaining types.

Case 2. ℓ is of type A_α, $\alpha \geq 3$.

We will show that in this case, $\ell \notin S(G)$. To do this we use Lemma 2.1.4 to show that ${}^U\mathcal{O}_\ell$ is not factorial. Then ${}^U\mathcal{O}_\ell$ is not regular, and by Lemma 2.1.2 $\mathcal{O}_\ell$ is not regular, so $\ell \notin S(G)$.

For ℓ of type A_α, $\alpha \geq 3$, choose $\mathcal{D}$ as prescribed in Chapter one. We construct $Z = Z(\mathcal{D},\mathcal{W}) = \operatorname{Spec} {}^U A(\mathcal{D},\mathcal{W})$. There is a point $z_0 \in Z$ such that $\mathcal{O}_{Z,z_0} = {}^U\mathcal{O}_\ell$. Let T be a maximal torus of B. Then T acts on Z, and z_0 is fixed by this action. Then we can apply Lemma 2.1.4. The set of eigenvectors of T in $k(Z) = {}^U k(G)$ is simply $\mathcal{P}$. So suppose

$F \in \mathcal{O}_{Z,z_o} \cap P$ cannot be expressed as a non-trivial product of non-units in $\mathcal{O}_{Z,z_o} \cap P$; then by the Lemma F is in fact extremal in $\mathcal{O}_{Z,z_o}$. We will construct such an F with the additional property that the ideal generated by F is not prime. (We say simply that F is not prime.) So we will have shown that $\mathcal{O}_{Z,z_o}$ is not factorial.

It remains to construct F. Recall that $\mathcal{W} = \{w_1,\ldots,w_\alpha\}$ with $w_i = v(D_i,r_i)$, $r_i \in \mathbb{Q} \cap (-1,1]$ for $i = 1,\ldots,\alpha$, and $\mathcal{D} = \mathbb{P}^1 - \{D_1,\ldots,D_\alpha\}$. Denote $r_i = \frac{p_i}{q_i}$ with $p_i, q_i \in \mathbb{Z}$, $q_i > 0$ and p_i and q_i relatively prime.

Let $F = f_{D_1}^{-q_1} f_{D_2}^{-p_1}$.

<u>Claim</u> : $F \in \mathcal{O}_{Z,z_o} \cap P$ and is not prime in $\mathcal{O}_{Z,z_o}$.

<u>Proof of claim</u> : Obviously $F \in {}^{U}A(\mathcal{D}) \cap P$. We will show that $w_i(F) \geq 0$ for $i = 1,\ldots,\alpha$.

$$w_1(F) = -q_1 r_1 + p_1 = 0 ;$$
$$w_2(F) = q_1 - p_1 r_2 = q_1(1 - r_1 r_2) \geq 0;$$
$$w_i(F) = q_1 + p_1 = q_1(1+r_1) > 0 \quad \text{for } i = 3,\ldots,\alpha.$$

Therefore, $F \in {}^{U}A(\mathcal{D},\mathcal{W}) \subset \mathcal{O}_{Z,z_o}$, and the first part of the claim is proven.

Also note that F is a regular function on $Z(\mathcal{D},\mathcal{W})$, which is a normal variety. The zero set of F includes one or more of the codimension one subvarieties of $Z(\mathcal{D},\mathcal{W})$ corresponding to the valuations w_i, $i = 1,\ldots,\alpha$. Each of these subvarieties contains z_o. If $\alpha \geq 4$ or if $r_1 r_2 < 1$, this zero set contains at least two of these subvarieties and is therefore not irreducible. Therefore if $\alpha \geq 4$, if $r_1 < 1$, or if $r_2 < 1$, then F is not prime in $\mathcal{O}_{Z,z_o}$.

Now if $\alpha = 3$ and $r_1 = r_2 = 1$, then $F = (f_{D_1} f_{D_2})^{-1}$. If $r_3 > 0$, consider $F \cdot (f_{D_2} f_{D_3})^{-1} \cdot (f_{D_1} f_{D_3})^{-1} = (f_{D_1} f_{D_2} f_{D_3})^{-2}$; it is easy to check that each of the above terms belong to $\mathcal{O}_{Z,z_o}$, but F does not divide $(f_{D_1} f_{D_2} f_{D_3})^{-1}$; so F is not prime. If $r_3 < 0$, consider $F \; f_{D_1}^{-2} = (f_{D_1} f_{D_3})^{-1} \cdot (f_{D_2} f_{D_3})^{-1}$; again, each of the terms above belong to $\mathcal{O}_{Z,z_o}$, but F does not divide either of the terms on the right in $\mathcal{O}_{Z,z_o}$. So F is not prime. This finishes the proof of the claim.

Now we will show that F cannot be expressed as a non-trivial product of non-units in $\mathcal{O}_{Z,z_o}$. Suppose $F = g_1 g_2$ with $g_1, g_2 \in \mathcal{O}_{Z,z_o} \in \mathcal{P}$. We will show that either g_1 or g_2 is a unit in $\mathcal{O}_{Z,z_o}$. We write $g_1 = (f_{D_1})^m h$ with $h \in \mathcal{P}$ such that the order of f_{D_1} in h is zero. Since $h \in \mathcal{P}$, it is homogeneous (that is, $\{cf_D \mid c \in k^*, D \in {}^B\mathcal{D}(G)\}$ is identified with the set of linear polynomials in two variables; h being a product of elements of this type, it is a homogeneous polynomial of two variables). We set d equal to the degree of h.

Now $w_1(g_1) \geq 0$ means $mr_1 - d \geq 0$, and $w_1(g_2) \geq 0$ means that $mr_1 - d \leq 0$, since $w_1(F) = 0$. So $mr_1 = d$. I claim that the degrees of g_1 and g_2 must be negative. To show this, choose $v = v(\ ,-1)$: this valuation dominates ℓ and for $h \in \mathcal{P}$, $v(h) = -\deg h$; so g_1 and g_2 being in $\mathcal{O}_v \cap \mathcal{P}$, their degrees must be negative. This tells us that $\deg F \leq \deg g_1 \leq 0$, that is $q_1 + p_1 \geq -m-d \geq 0$; since $mr_1 = d$, it follows that $q_1(1+r_1) \geq -m(1+r_1) \geq 0$, i.e., $q_1 \geq -m \geq 0$. Also since $mr_1 \in \mathbb{Z}$, we know that q_1 divides m, so either $m = 0$ or $-q_1$; this means that either $\deg g_1 = 0$ or $\deg g_1 = \deg F$. If $\deg g_1 = 0$, then $g_1 \in \mathcal{O}_v^*$, so it is a unit in $\mathcal{O}_{Z,z_o}$; on the other hand, if $\deg g_1 = \deg F$, then $\deg g_2 = 0$ and therefore g_2 is a unit in $\mathcal{O}_{Z,z_o}$.

So by Lemma 2.1.4, F is extremal in $\mathcal{O}_{Z,z_o}$. So we conclude that ${}^U\mathcal{O}_\ell = \mathcal{O}_{Z,z_o}$ is not factorial. Therefore $\ell \notin S(G)$.

This completes the proof of the proposition.

□

Remark : For this case of $\Gamma = \{e\}$, one can replace Lemma 2.1.2 by a stronger Lemma which says that in fact $A(\mathcal{D},\mathcal{W}) \cong k[U] \otimes {}^U A(\mathcal{D},\mathcal{W})$ when $\mathcal{D} \neq \mathbb{P}^1$. Then, for example if X is a smooth embedding, we know how to cover X with charts each isomorphic to $\mathbb{A}_k^3$: for all smooth localities we can choose $\mathcal{D}$ and $\mathcal{W}$ such that $Z(\mathcal{D},\mathcal{W}) = \text{Spec } {}^U A(\mathcal{D},\mathcal{W})$ is a torus embedding; it is smooth and of dimension two, so then $Z(\mathcal{D},\mathcal{W}) \cong \mathbb{A}_k^2$, and by the new Lemma, Spec $A(\mathcal{D},\mathcal{W}) \cong \mathbb{A}_k^3$. Because this result is stronger and the method of proof is interesting, we will state and prove this Lemma.

As usual, we have $G = SL(2,k)$, B a Borel subgroup of G and U the unipotent radical of B. We fix another Borel subgroup B^- opposed to B. Suppose G' is an open B-stable subvariety of $BB^- = UB^-$. Then $G' \cong U \times V$ where V is an open affine subvariety of B^-. Therefore $k[G'] \cong k[U] \otimes k[V]$. This is a U-isomorphism, so $k[V] \cong {}^U k[G']$; that is $k[G'] \cong k[U] \otimes {}^U k[G']$.

Lemma 2.2.2. Suppose G' is an open subvariety of BB^-, and suppose $w_1,\dots,w_\alpha \in \mathcal{V}(G)$. Let $A = k[G'] \cap \mathcal{O}_{w_1} \cap \dots \cap \mathcal{O}_{w_\alpha}$. Then the isomorphism $k[G'] \cong k[U] \otimes {}^U k[G']$ induces an isomorphism $A \cong k[U] \otimes {}^U A$.

To apply this lemma to our case, fix $D_o = Bg \in {}^B\mathcal{D}(G)$ where $B^- = g^{-1} B g$. Given ℓ a locality of a one-dimensional orbit such that $\mathcal{O}_\ell$ is a localization of $A(\mathcal{D},\mathcal{W})$, we have

$\mathcal{D} \neq \mathbb{P}^1$. Then, by possibly changing the base point, we can assume that $D_o \notin \mathcal{D}$. Now we are in the situation of the Lemma : $A(\mathcal{D})$ is the coordinate ring of a B-stable subvariety of $BB^- = G-\{D_o\}$, and the Lemma shows that $A(\mathcal{D},\mathcal{W}) \cong k[U] \otimes {}^UA(\mathcal{D},\mathcal{W})$.

<u>Proof of Lemma.</u>

We have

$$\begin{array}{ccc} BB^- & \cong & U \times B^- \\ \cup & & \cup \\ G' & \cong & U \times V \end{array}$$

So

$$\begin{array}{ccc} k[G'] & \cong & k[U] \otimes {}^Uk[G'] \\ \cup & & \cup \\ A & & k[U] \otimes {}^UA \ . \end{array}$$

We will show that the isomorphism above induces an isomorphism of A onto $k[U] \otimes {}^UA$.

Suppose that $f \in k[G']$ and the image of f under the isomorphism above in shortest form is $\sum_{i=1}^{m} g_i \otimes f_i$ with $g_i \in k[U]$, $f_i \in {}^Uk[G']$, $i = 1,\ldots,m$. Let $w \in \mathcal{V}(G)$. Once we prove that $w(f) = \inf_i w(f_i)$, the lemma is proven.

First note that it is enough to show for $f \in k[BB^-]$ that $w(f) = \inf w(f_i)$. For if $f' \in k[G']$, then there exists $h \in {}^Uk[G']$ such that $f = f'h \in k[BB^-]$. If the image of f is $\sum_{i=1}^{m} g_i \otimes f_i$, then the image of f' is $\Sigma g_i \otimes (f_i/h)$; also $w(f') = \inf_i w(f_i/h)$ if and only if $w(f) = \inf_i w(f_i)$.

We fix some notation :

$$U = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in k\} ,$$

$$B^- = \{\begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix} \mid x \in k^*,\ y \in k\},$$

and $$G = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11}a_{22} - a_{12}a_{21} = 1\} .$$

Note that $k[BB^-] = k[G]_{a_{22}} = k[a_{21}, a_{22}, \frac{a_{12}}{a_{22}}]_{a_{22}}$.

The isomorphism $k[U \times B^-] \xrightarrow{\sim} k[BB^-]$ is defined by

$$x \longrightarrow a_{22}^{-1}$$

$$y \longrightarrow a_{21}$$

and $$a \longrightarrow \frac{a_{12}}{a_{22}} .$$

So the isomorphism $k[U] \otimes {}^{U}k[G'] \xrightarrow{\sim} k[G']$ is defined by

$$a \otimes 1 \longrightarrow \frac{a_{12}}{a_{22}}$$

and $$1 \otimes f \longrightarrow f .$$

Let $f \in k[BB^-]$. Then there exists $N \in \mathbb{N}$ such that $a_{22}^{N}f$ is a polynomial function of a_{21}, a_{22}, and $\frac{a_{12}}{a_{22}}$:

$$a_{22}{}^{N}f = \sum_{i=o}^{n} \left(\frac{a_{12}}{a_{22}}\right)^{i} f_i(a_{21}, a_{22}) .$$

The image of $a_{22}{}^{N}f$ in $k[U] \otimes {}^{U}k[G']$ is

$$\sum_{i=o}^{n} a^{i} \otimes f_i(a_{21}, a_{22}) .$$

We will show that $w(a_{22}{}^{N}f) = \inf_i w(f_i(a_{21}, a_{22}))$; then since $a_{22}{}^{N} \in {}^{U}k[G']$, the lemma will be proven.

We know that any $w \in V(G)$ is induced by the germ of a formally divergent curve, $\lambda(t)$ (see § 4 [9]). Also we can choose

$$\lambda(t) = \begin{pmatrix} t^{p} & 0 \\ t^{q} & t^{-p} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

with $p, q \in \mathbb{Z}$ and $u \in k$ [9]. Let i_λ be the morphism

$$i_\lambda : k[G] \xrightarrow{\text{comult.}} k[G] \otimes k[G] \xrightarrow{1 \otimes \lambda} k[G] \otimes k((t)) \longrightarrow k[G]((t)).$$

Then for $g \in k[G]$, $w(g)$ is the order of t in $i_\lambda(g)$. We have

$$i_\lambda(a_{22}) = (1 \otimes \lambda)(a_{21} \otimes a_{12} + a_{22} \otimes a_{22}) = a_{22} \otimes t^{-p} ;$$

$$i_\lambda(a_{12}) = (1 \otimes \lambda)(a_{11} \otimes a_{12} + a_{12} \otimes a_{22}) = a_{12} \otimes t^{-p} ;$$

$$i_\lambda(a_{21}) = (1 \otimes \lambda)(a_{21} \otimes a_{11} + a_{22} \otimes a_{21}) = a_{21} \otimes t^{p} + a_{22} \otimes (t^{q} + ut^{-p}).$$

Now $a_{22}{}^{N+n}f \in k[G]$, so $w(a_{22}{}^{N+n}f)$ is the order of t in $i_\lambda(a_{22}{}^{N+n}f)$. Now

$$i_\lambda(a_{22}{}^{N+n}f) = \sum_{j=o}^{n} i_\lambda(a_{12}{}^{j}) \cdot i_\lambda(a_{22}{}^{n-j} f_j(a_{21},a_{22}))$$

$$= \sum_{j=o}^{n} (a_{12}{}^{j} \otimes t^{-pj}) i_\lambda(a_{22}{}^{n-j} f_j(a_{21},a_{22})).$$

Let us denote by ord_t the order of t. For each j, note that $i_\lambda(a_{22}{}^{n-j}f_j(a_{21},a_{22}))$ does not depend on a_{12}. Therefore, we have

$$\mathrm{ord}_t(i_\lambda(a_{22}{}^{N+n}f)) = \inf_j \{-pj + \mathrm{ord}_t(i_\lambda(a_{22}{}^{n-j}f_j(a_{21},a_{22}))\} .$$

Now $i_\lambda(a_{22}) = a_{22} \otimes t^{-p}$, so

$$\mathrm{ord}_t(i_\lambda(a_{22}{}^{N+n}f)) = \inf_j\{-pj - p(n-j) + \mathrm{ord}_t(f_j(a_{21},a_{22}))\}$$

$$= -pn + \inf_j\{\mathrm{ord}_t\, f_j(a_{21},a_{12})\} .$$

So $w(a_{22}{}^{N+n}f) = -pn + \inf_j\{w(f_j(a_{21},a_{12}))\}$, and $w(a_{22}) = -p$, so $w(a_{22}{}^{N}f) = \inf_j\{w(f_j(a_{21},a_{22}))\}$. This finishes the proof of the lemma.

□

§ 3. Case of $\Gamma = \mathbb{Z}/2\mathbb{Z}$.

Let Γ be the group of two elements. Then as a subgroup of G, $\Gamma = \{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\}$. So $G/\Gamma = PGL_2(k)$. In this section we calculate the smooth embeddings of $PGL_2(k)$.

Recall that ${}^{B}\mathcal{D}(G/\Gamma) = {}^{B}\mathcal{D}(G) \cong \mathbb{P}^1$, and the diagram of $\mathcal{V}(G/\Gamma)$ is the same as the diagram of $\mathcal{V}(G)$. However the two cases are not the same : when $\Gamma = \{e\}$ then $f_D = g_D$, but for $\Gamma = \mathbb{Z}/2\mathbb{Z}$ we have $f_D = g_D^2$. This changes the calculation of $\mathcal{S}(G/\Gamma)$.

Proposition 2.3.1. Let $\ell \in L_1^n(G/\Gamma)$, where $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Then $\ell \in \mathcal{S}(G/\Gamma)$ if and only if it is one of the following types :

1a) Type A_1 with $r_1 = -\dfrac{1}{2n+1}$, $n \in \mathbb{N}^+$;

1b) Type A_2 with $r_i = \dfrac{p_i}{q_i}$ and either

(i) $q_1q_2-p_1p_2 = 1$ and 2 divides $(q-p_1)(q_2-p_2)$

or (ii) $q_2q_1-p_1p_2 = 2$ and 2 divides both q_1-p_1 and q_2-p_2;

1c) Type A_3 with $r_1 = r_2 = r_3 = 1$;

2) Type AB with $r_i = \dfrac{p_i}{q_i}$ and either

(i) $|p_1q_2-p_2q_1| = 1$ and 2 divides $(q_1-p_1)(q_2-p_2)$

or (ii) $|p_1q_2-p_2q_1| = 2$ and 2 divides both q_1-p_1 and q_2-p_2;

3) Type B_+ with $r_1 = -1$;

4) Type B_- with $r_1 = \dfrac{1}{2n+1}$, $n \in \mathbb{N}^+$;

5) Type C.

(As in section 2, if we write $r = \frac{p}{q}$, we choose $q > 0$ and $p,q \in \mathbb{Z}$ and relatively prime.)

Proof.

The proof of this proposition follows the same scheme as the proof of Proposition 2.2.1.

If ℓ is of Type C then $\ell \in S(G/\Gamma)$, and if ℓ is of Type B_o, then $\ell \notin S(G/\Gamma)$. (Lemma 2.1.1).

In all other cases, ℓ is the locality of an orbit of dimension one. As in section two, we first consider the case where we can apply the theory of torus embeddings.

Case 1 : ℓ is of type AB, B_+, B_-, A_1 or A_2.

In each of these cases, one can choose $\mathcal{D}$ such that $Z(\mathcal{D},\mathcal{W}) = \text{Spec}\ {}^{U}A(\mathcal{D},\mathcal{W})$ is a torus embedding by Lemma 2.1.2. Then by Lemma 2.1.6, $\ell \in S(G/\Gamma)$ if and only if $Z(\mathcal{D},\mathcal{W})$ is smooth.

The ring of regular functions of the torus for which $Z(\mathcal{D},\mathcal{W})$ is an embedding is $A(\mathbb{P}^1 - \{D_1,D_2\}) = k[f_{D_1}, f_{D_1}^{-1}, g_{D_1}g_{D_2}, (g_{D_1}g_{D_2})^{-1}]$. So if we call the torus S, then $X(S) = \{(f_{D_1})^{m_1}(g_{D_1}g_{D_2})^{m_2} \mid m_1,m_2 \quad \mathbb{Z}\}$. Now for each type we must find the primitive vectors e_1 and e_2 in $X_*(S)$ which generate the sector of linear functionals which are positive on $X(S) \cap {}^{U}A(\mathcal{D},\mathcal{W})$ and then calculate the determinant, $\det(e_1 \quad e_2)$. Then $\ell \in S(G/\Gamma)$ if and only if $\det(e_1 \quad e_2) = \pm 1$.

Type AB. Let $w_1 = v(D_1,r_1)$ and $w_2 = v(D_1,r_2)$ with $r_i = \frac{p_i}{q_i}$, $i = 1,2$. We choose $\mathcal{D} = \mathbb{P}^1 - \{D_1,D_2\}$ for an arbitrary $D_2 \neq D_1$.

Now $w_i(f_{D_1}^{\ m_1}(g_{D_1}g_{D_2})^{m_2}) = (m_1 + \frac{m_2}{2})r_i - \frac{m_2}{2}$. This is positive if and only if $(2p_i)m_1 + (p_i-q_i)m_2 \geq 0$. So

$$X(S) \cap {}^{U}A(\mathcal{D},W) = \{f_{D_1}^{\ m_1}(g_{D_1}g_{D_2})^{m_2} \mid 2p_im_1 + (p_i-q_i)m_2 \geq 0, i = 1,2\}.$$

Let $e_1' = (2p_1, p_1-q_1)$ and $e_2' = (2p_2, p_2-q_2)$. Then e_1' and e_2' generate the set of linear functionals positive on

$X(S) \cap {}^{U}A(\mathcal{D},\mathcal{W})$, but they may not be primitive. Now $\det(e_1' \ e_2') = 2(q_1p_2-q_2p_1)$. Note that e_i' is primitive if and only if 2 does not divide q_i-p_i for $i=1,2$. Now it is easy to see that the condition given in the proposition is given such that the determinant of the primitive vectors is ± 1.

Type B_+. Let $w_1 = v(D_1,r_1)$, $-1 \le r_1 = \frac{p_1}{q_1} < 1$. We choose $\mathcal{D} = \mathbb{P}^1-\{D_2\}$ for an arbitrary $D_2 \ne D_1$. Then for $f_{D_1}^{m_1}(g_{D_1}g_{D_2})^{m_2}$ to be in ${}^{U}A(\mathcal{D},\mathcal{W})$, it must be in $A(\mathcal{D})$, and we must have $w_1(f_{D_1}^{m_1}(g_{D_1}g_{D_2})^{m_2}) \ge 0$. So

$$X(S) \cap {}^{U}A(\mathcal{D},\mathcal{W}) = \{f_{D_1}^{m_1}(g_{D_1}g_{D_2})^{m_2} \mid 2m_1+m_2 \ge 0 \text{ and }$$

$$2p_1m_1 + (p_1-q_1)m_2 \ge 0\}.$$ Let $e_1' = (2,1)$ and $e_2' = (2p_1,p_1-q_1)$. Then $\det(e_1' \ e_2') = -2q_1$. Now e_1' is a primitive vector. So for $Z(\mathcal{D},\mathcal{W})$ to be smooth, we must have that e_2' is not primitive; therefore we must have that 2 divides p_1-q_1. Also it is necessary that $q_1 = 1$. Since $-q_1 \le p_1 < q_1$, this means that $r_1 = -1$. So $\ell \in S(G/\Gamma)$ if and only if $r_1 = -1$.

Type B_-. Let $w_1 = v(D_1,r_1)$ with $0 < r_1 = \frac{p_1}{q_1} < 1$. We must choose $\mathcal{D} = \mathbb{P}^1 - \{D_1\}$. Pick an arbitrary $D_2 \in \mathcal{D}$. Then

$$X(S) \cap {}^{U}A(\mathcal{D},W) = \{f_{D_1}^{m_1}(g_{D_1}g_{D_2})^{m_2} : m_2 \ge 0 \text{ and }$$

$$2p_1m_1 + (p_1-q_1)m_2 \ge 0\}.$$

Let $e_1' = (0,1)$ and $e_2' = (2p_1,p_1-q_1)$. Then $\det(e_1' \ e_2') = -2p_1$. Now e_1' is primitive, so $Z(\mathcal{D},\mathcal{W})$ is smooth if and only if 2 divides p_1-q_1 and $p_1 = 1$. (Remember that in this case p_1 is always positive). That is, $\ell \in S(G/\Gamma)$ if and only if $r_1 = \frac{1}{2n+1}$, $n \in \mathbb{N}^+$.

Type A_1. This type is identical to the type B_- except that we must choose $-1 < r_1 = \frac{p_i}{q_i} < 0$. So we find that $Z(\mathcal{D},\mathcal{W})$ is smooth if and only if $p_1 = -1$ and 2 divides $p_1 - q_1$. That is, $\ell \in S(G/\Gamma)$ if and only if $r_1 = \frac{-1}{2n+1}$, $n \in \mathbb{N}^+$.

Type A_2. Let $w_1 = v(D_1,r_1)$ and $w_2 = v(D_2,r_2)$ with $r_i = \frac{p_i}{q_i}$, $i = 1,2$. Then we must choose $\mathcal{D} = \mathbb{P}^1 - \{D_1,D_2\}$. Now
$X(S) \cap {}^U A(\mathcal{D},\mathcal{W}) = \{f_{D_1}^{m_1}(g_{D_1}g_{D_2})^{m_2} \mid 2p_1m_1 + (p_1-q_1)m_2 \geq 0$ and $-2q_2m_1 + (p_2-q_2)m_2 \geq 0\}$. Let $e_1' = (2p_1, p_1-q_1)$ and $e_2' = (-2q_2, p_2-q_2)$. Then $\det(e_1'\ e_2') = -2(q_1q_2 - p_1p_2)$. Now it is easy to see that $\ell \in S(G/\Gamma)$ if and only if one of the conditions given in the proposition is satisfied.

So the proposition is proven for these types.

Now we must check the proposition for the case when is of type A_α, $\alpha \geq 3$. We have $\mathcal{W} = \{w_1,\ldots,w_\alpha\}$ with $w_i = v(D_i,r_i)$ and $\mathcal{D} = \mathbb{P}^1 - \{D_1,\ldots,D_\alpha\}$. We can suppose $r_1 \leq r_i$, $i = 2,\ldots,\alpha$.

Case 2. ℓ is of type A_α, $\alpha \geq 4$ or of type A_3 with $r_1 < 1$. This case is very similar to the case 2 of Proposition 2.2.1. To show $\ell \notin S(G/\Gamma)$ by Lemma 2.1.2 it is enough to show ${}^U\mathcal{O}_\ell$ is not regular. We show ${}^U\mathcal{O}_\ell$ is not factorial, so not regular. Again we construct an F which we show is extremal by Lemma 2.1.4 but which is not prime. Recall that for the case when $\Gamma = \{e\}$, to show that the F we constructed was not prime for $\alpha = 3$ when $r_1 = r_2 = 1$, we had to use an extra argument. This argument does not work for $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Other than that, the proof is quite similar to the case $\Gamma = \{e\}$.

Let $r_1 = \frac{p_1}{q_1}$ with $p_1, q_1 \in \mathbb{Z}$, $q_1 > 0$, and p_1 and q_1 relatively prime.

(i) Suppose $p_1 + q_1$ is even.

Then I define $F := \dfrac{1}{g_{D_1}^{q_1} g_{D_2}^{p_1}}$. Clearly $F \in \mathcal{P}^{(\Gamma)} \cap A(\mathcal{D})$.

Also we have
$$w_1(F) = \frac{-q_1 r_1 + p_1}{2} = 0$$
$$w_2(F) = \frac{q_1 - p_1 r_2}{2} = \frac{q_1}{2}(1 - r_1 r_2) \geq 0$$
and > 0 if $r_1 < 1$
$$w_i(F) = \frac{q_1 + p_1}{2} = \frac{q_1}{2}(1 + r_1) > 0, \quad i = 3, \ldots, \alpha.$$

So we see that $F \in {}^U A(\mathcal{D}, \mathcal{W})$. Let $Z = Z(\mathcal{D}, \mathcal{W}) = \mathrm{Spec}\, {}^U A(\mathcal{D}, \mathcal{W})$, and let z_o be the point of Z such that $\mathcal{O}_{Z,z} = {}^U\mathcal{O}_\ell$. Now F is not prime in $\mathcal{O}_{Z,z_o}$, but F is extremal in $\mathcal{O}_{Z,z_o}$. The proof is identical to the proof for $\Gamma = \{e\}$. So in this case $\mathcal{O}_{Z,z_o}$ cannot be factorial.

(ii) Suppose $p_1 + q_1$ is odd.

Then $F = \dfrac{1}{g_{D_1}^{q_1} g_{D_2}^{p_1}}$ is not in $k(G)^\Gamma$. So let $F' = F^2$. Now we have $F' \in \mathcal{P}^{(\Gamma)} \cap \mathcal{O}_{Z,z_o}$ and F' is not prime. We will now show it is irreducible in $\mathcal{O}_{Z,z_o}$. Suppose not. Then by Lemma 2.1.4, it can be factored in $\mathcal{P}^{(\Gamma)} \cap \mathcal{O}_{Z,z_o}$. Suppose

$F' = g_1 g_2$ with $g_1, g_2 \in \mathcal{O}_{Z,z_o} \cap \mathcal{P}^{(\Gamma)}$. We must show that either g_1 or g_2 is a unit. Suppose $g_1 = (g_{D_1})^m h$ where the order of g_{D_1} in h is 0. Since $h \in \mathcal{P}^{(\Gamma)}$, h is homogeneous. Let $d = \deg h$. The degree of g_{D_1} is 1. Since $g_1 \in k(G)^\Gamma$, the degree of g_1 must be even; therefore $m+d$ is even. Using the same method as in the case $\Gamma = \{e\}$, we find $mr_1 = d$

and deg $F' \leq \deg g_1 \leq 0$. That is $2(q_1+p_1) \geq -m-d \geq 0$, or $2q_1 \geq -m \geq 0$. Also q_1 divides m. If $m = 0$, then $d = 0$ and $g_1 \in \mathcal{O}^*_{Z,z_o}$. If $m = -2q_1$, then $q_2 \in \mathcal{O}^*_{Z,z_o}$. Finally, if $m = -q_1$, then $d = -p_1$, and m+d is odd, which is impossible. So F' is extremal but not prime. This contradicts the hypothesis that $\mathcal{O}_{Z,z_o} = {}^U\mathcal{O}_\ell$ is factorial.

So the proposition is proven in this case.

Now we have one case left to do.

Case 3. ℓ is of Type A_3 with $r_1 = r_2 = r_3 = 1$.

We will show that $\ell \in S(G/\Gamma)$ by proving that ${}^U\mathcal{O}_\ell$ is regular : we will find two elements of $m_\ell \cap {}^UA(\mathcal{D},\mathcal{W})$ that generate $m_\ell \cap {}^UA(\mathcal{D},\mathcal{W})$.

Let $X = \dfrac{1}{g_{D_1}g_{D_2}}$, $Y = \dfrac{1}{g_{D_1}g_{D_3}}$, $Z = \dfrac{1}{g_{D_2}g_{D_3}}$. Note that since $g_{D_1} = g_{D_2} + ag_{D_3}$ for some $a \in k^*$, we have that $Z = Y + aX$.

Claim : X and Y generate the ideal $m_\ell \cap {}^UA(\mathcal{D},\ )$.

Proof : Clearly, $(X,Y) \subset m_\ell \cap {}^UA(\mathcal{D},\mathcal{W})$. By Lemma 2.1.3 we only need to show that any element of $m_\ell \cap {}^UA(\mathcal{D},\mathcal{W}) \cap P^{(\Gamma)}$ is in the ideal (X,Y). Suppose $F \in m_\ell \cap {}^UA(\mathcal{D},\mathcal{W}) \cap P^{(\Gamma)}$.

Then

$$F = cg_{D_1}^{m_1} g_{D_2}^{m_2} g_{D_3}^{m_3} \prod_{i=1}^{d} (g_{D_2}+a_i g_{D_3}) \quad c \in k^*,\ a_i \in k^*$$

$$i = 1,\dots,d$$

with

$$2w_1(F) = m_1-m_2-m_3-d \geq 0,$$

$$2w_2(F) = m_2-m_1-m_3-d \geq 0,$$

$$2w_3(F) = m_3-m_1-m_2-d \geq 0,$$

and $\qquad m_1+m_2+m_3+d \quad$ is even.

$$\text{Then } F = c g_{D_1}^{m_1+d} g_{D_2}^{m_2+d} g_{D_3}^{m_3+d} \prod_{i=1}^{d} (Y + a_i X)$$

$$= X^{\alpha} Y^{\beta} Z^{\gamma} \prod_{i=1}^{d} (Y + a_i X)$$

where $\alpha = \dfrac{m_3-m_1-m_2-d}{2} \geq 0$ and $\alpha \in \mathbb{Z}$,

$\beta = \dfrac{m_2-m_1-m_3-d}{2} \geq 0$ and $\beta \in \mathbb{Z}$,

and $\gamma = \dfrac{m_1-m_2-m_3-d}{2} \geq 0$ and $\gamma \in \mathbb{Z}$.

So $F = cX^{\alpha}Y^{\beta}(Y+aX)^{\gamma} \prod_{i=1}^{d} (Y+a_iX) \in (X,Y)$, and the claim is proven.

Therefore, in this case $\ell \in S(G/\Gamma)$. This finishes the proof of the proposition.

□

§ 4. Other examples.

In this section, we will give a few examples of how to use the methods described in section one to calculate $S(G/\Gamma)$ for Γ other than $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$.

Fix Γ a finite subgroup of order greater than or equal to three. Denote by $\overline{\Gamma}$ the intersection of Γ with the center of G, and set $n = [\Gamma:\overline{\Gamma}]$ (so if the order $|\Gamma|$ of Γ is odd, then $n = |\Gamma|$, and if $|\Gamma|$ is even, then $n = |\Gamma|/2$).

Let $D \in \mathbb{P}^1/\Gamma$ such that $b(D)$ is minimal. Note that since $b(D) \leq 0$, there is no type B_- or B_o locality with V_ℓ of the form $\{v(D,r)\}$. Let $\ell \in L_1^n(G/\Gamma)$ be either of type B_+ with $V_\ell = \{v(D,r)\}$, $-1 \leq r < b(D)$ or of type AB with

$V_\ell = \{v(D,r_1), v(D,r_2)\}$, $-1 \le r_1 < r_2 \le b(D)$; then one can use Lemma 2.1.5 together with the results of sections 2 and 3 to check when $\ell \in S(G/\Gamma)$. To explain how to do this we fix some notation.

Consider the set $\tilde{D} = \{D_1,\ldots,D_n\} \subset {}^B\mathcal{D}(G/\bar{\Gamma})$ of irreducible components of the inverse image of D by the projection $G/\bar{\Gamma} \to G/\Gamma$. For each $i = 1,\ldots,n$ we choose $\tilde{\ell}_i \in L_1^n(G/\bar{\Gamma})$ as follows. Let $\tilde{\ell}_i$ be of the same type as ℓ, and such that the elements of $V_{\tilde{\ell}_i}$ are the elements of $V(G/\bar{\Gamma}) = V(G/\{e\})$ of the form $v(D_i,s)$ which are extensions of elements of V_ℓ.

<u>Proposition 2.4.1</u>. Suppose $\ell \in L_1^n(G/\Gamma)$ is of type B_+ with $V_\ell = \{v(D,r)\}$ or of type AB with $V_\ell = \{v(D,r_1), v(D,r_2)\}$ and $b(D)$ is minimal. Then $\ell \in S(G/\Gamma)$ if and only if $\tilde{\ell}_i \in S(G/\bar{\Gamma})$ for any i.

<u>Proof</u>.

This proposition is a consequence of Lemma 2.1.5. Choose $\mathcal{D} \subset {}^B\mathcal{D}(G/\Gamma)$ cofinite such that $\mathcal{O}_\ell$ is a localization of $A(\mathcal{D},V_\ell)$, and choose $v \in F_\ell$; then v is of the form $v(D,s)$, $-1 < s \le b(D)$. We denote by $\tilde{v}_i$ the extension in $V(G/\bar{\Gamma})$ of the form $V(D_i,s_i)$ with $-1 < s_i \le 1$, $i = 1,\ldots,n$. Let $\tilde{\mathcal{D}}$ be the inverse image of $\mathcal{D}$ in the projection $\mathbb{P}^1 \to \mathbb{P}^1/\Gamma$ and $\tilde{V}_\ell$ be the set of extensions of elements of V_ℓ in $V(G/\bar{\Gamma})$.

Since $A(\mathcal{D},V_\ell)$ is a Krull ring, its integral closure in $k(G/\bar{\Gamma})$ is also a Krull ring, and its essential valuations are all the extensions of essential valuations of $A(\mathcal{D},V_\ell)$ [2]; so it is $A(\tilde{\mathcal{D}},\tilde{V}_\ell)$. The localization of $A(\tilde{\mathcal{D}},\tilde{V}_\ell)$ in the ideal $m_{\tilde{v}_i} \cap A(\tilde{\mathcal{D}},\tilde{V}_\ell)$ is the local ring of $\ell_i' \in L_1^n(G/\bar{\Gamma})$; in fact $\ell_i' = \tilde{\ell}_i$; to see this note that $\mathcal{O}_{\tilde{\ell}_i}$ is the localization of $A(\tilde{\mathcal{D}},V_{\tilde{\ell}_i})$ in the ideal $m_{\tilde{v}_i} \cap A(\tilde{\mathcal{D}},V_{\tilde{\ell}_i})$ and $A(\tilde{\mathcal{D}},\tilde{V}_\ell) \subset A(\mathcal{D},V_{\tilde{\ell}_i}) \subset \mathcal{O}_{\ell_i'}$. Now one can show that $\mathcal{O}_{\tilde{\ell}_i}$ is a localization of the integral closure of $\mathcal{O}_\ell$, so we can apply Lemma 2.1.5. This finishes the proof of the proposition.

□

Now $\overline{\Gamma}$ is either $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$; so we know $S(G/\overline{\Gamma})$ from sections 2 and 3.

As before let Γ be a finite subgroup of G of order greater than or equal to 3.

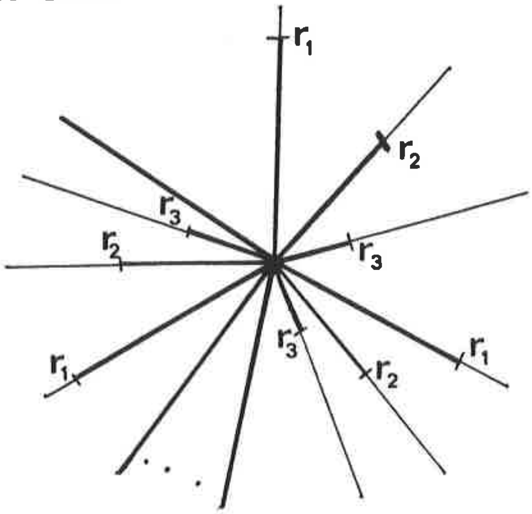
Proposition 2.4.2. Suppose $\ell \in L_1^n(G/\Gamma)$ is of type A_α such that $V_\ell = \{v(D_i,r_i)\}_{i=1,\ldots,\alpha}$ with $b(D_i)$ minimal for $i=1,\ldots,\alpha$. Then $\ell \notin S(G/\Gamma)$.

One can prove this proposition using methods similar to the ones used for type A_α, $\alpha \geq 3$ in sections 2 and 3. This is quite long, and one must study many cases separately. So we will instead give another proof, which uses the theorem of purity of Zariski : if Y is a normal variety and X is smooth and $Y \to X$ is a finite morphism, then the set of branch points is either empty or purely of codimension one [14].

Proof of the Proposition.

Let X be the minimal embedding with locality ℓ; that is X is an embedding with $\alpha+2$ orbits : α orbits of type C, one orbit of type A_α, and the open orbit. The diagram of X is of the form

We will find an embedding Y of $G/\overline{\Gamma}$ such that there is a finite morphism $Y \to X$ which is branched just over the orbit with locality ℓ. First we define Y. We use the notation given in the proof of the previous proposition. Let $\tilde{\ell} \in L_1^n(G/\overline{\Gamma})$ be the locality of Type $A_{n\alpha}$ with $\mathcal{V}_{\tilde{\ell}} = \tilde{\mathcal{V}}_{\ell}$. Then let Y be the minimal embedding with locality $\tilde{\ell}$. The diagram of Y is of the form



We will show that there is a finite morphism $Y \to X$. Now Spec $A(\mathcal{D}_\ell, \mathcal{V}_\ell)$ is an open subset of X and Spec $A(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{V}}_\ell)$ is an open subset of Y. The inclusion $A(\mathcal{D}_\ell, \mathcal{V}_\ell) = A(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{V}}_\ell)^\Gamma \subset A(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{V}}_\ell)$ induces a finite morphism Spec $A(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{V}}_\ell) \to$ Spec $A(\mathcal{D}_\ell, \mathcal{V}_\ell)$. Now $X = G \cdot$ Spec $A(\mathcal{D}_\ell, \mathcal{V}_\ell)$ and $Y = G \cdot$ Spec $A(\tilde{\mathcal{D}}_\ell, \tilde{\mathcal{V}}_\ell)$. The finite morphism above extends to a finite G-morphism $Y \to X$. It is clear that this morphism is unramified over the open orbit. Also, for any orbit T of type C of X there are n orbits of type C of Y which lie over T (this is because we chose $b(D_i)$ minimal for $i = 1, \ldots, \alpha$), so the morphism is unramified over T. Over the orbit of type A_α is the orbit of type $A_{n\alpha}$ of Y. So the set of branch points of $Y \to X$ is the orbit of type A_α; the orbit is of codimension two, so by the theorem of purity of Zariski, X is not smooth. We know X is smooth at every point not in the orbit with locality ℓ, so $\ell \notin S(G/\Gamma)$.

□

Now suppose Γ is cyclic of order greater than or equal to 3. Choose D_1 and D_2 of $\mathbb{P}^1/\Gamma$ such that $b(D_1) = b(D_2) = 1$. That is, D_1 and D_2 are the two fixed elements of $\mathbb{P}^1/\Gamma$. Suppose $\ell \in L_1^n(G/\Gamma)$ is of type A_1, A_2, AB, B_+ or B_- with each element of $\mathcal{W}$ of the form $v(D_i, r)$, $i = 1$ or 2. Then we can apply the theory of torus embeddings. The calculations are very similar to those in section 3. In this case, the ring of regular functions on the torus is $A(\mathbb{P}^1/\Gamma - \{D_1, D_2\}) = k[f_{D_1}, f_{D_1}^{-1}, g_{D_1}g_{D_2}, (g_{D_1}g_{D_2})^{-1}]$, so the characters of the torus are $\{f_{D_1}^{m_1}(g_{D_1}g_{D_2})^{m_2} : m_1, m_2 \in \mathbb{Z}\}$.

To give one example, we will prove the following proposition.

<u>Proposition 2.4.3</u>. Suppose Γ is cyclic of order greater than or equal to 3, and $\ell \in L_1^n(G/\Gamma)$ is of the type B_+ with $\mathcal{W} = \{v(D_1, r_1)\}$ and $b(D_1) = 1$. Then $\ell \notin S(G/\Gamma)$.

<u>Proof</u>.

We choose $\mathcal{D} = \mathbb{P}^1/\Gamma - \{D_2\}$ for an arbitrary $D_2 \neq D_1$. Now $-1 \le r_1 = \dfrac{p_1}{q_1} < 1$. By Lemma 2.1.6, $\ell \in S(G/\Gamma)$ if and only if $Z(\mathcal{D}, \mathcal{W})$ is a smooth torus embedding. The set of characters of the torus which are in ${}^{U}A(\mathcal{D}, \mathcal{W})$ is

$$\{f_{D_1}^{m_1}(g_{D_1}g_{D_2})^{m_2} \mid |\Gamma|m_1 + m_2 \ge 0 \quad \text{and} \quad |\Gamma|p_1m_1 + (p_1 - q_1)m_2 \ge 0\}.$$

Then let $e_1' = (|\Gamma|, 1)$ and $e_2' = (|\Gamma|p_1, p_1 - q_1)$. Then $\det(e_1' \;\; e_2') = -|\Gamma|q_1$. For $Z(\mathcal{D}, \mathcal{W})$ to be smooth, it would be necessary that $q_1 = 1$ and that $\dfrac{e_2'}{|\Gamma|} \in \mathbb{Z}^2$. That is $|\Gamma|$ divides $p_1 - q_1$. If $q_1 = 1$, then $r_1 = 0$ or -1, so $p_1 = 0$ or -1. Since $|\Gamma| \ge 3$, $|\Gamma|$ does not divide 1 or 2. So $Z(\mathcal{D}, \mathcal{W})$ is never smooth, and therefore $\ell \notin S(G/\Gamma)$.

□

CHAPTER III: SMOOTH COMPLETE B/Γ-EMBEDDINGS: A GEOMETRIC APPROACH

In the previous two chapters, we study $SL(2,k)/\Gamma$-embeddings in a combinatorical way. We describe an embedding by giving a set of data which defines the local rings of the orbits. A disadvantage to this approach is that one does not see the geometry of the varieties obtained.

In this chapter we will start developping a more geometric way of looking at some of these embeddings. The idea is as follows. Consider the case where $\Gamma \subset SL(2,k)$ is a finite cyclic group. Choose B a Borel subgroup of SL(2,k) containing Γ. Suppose we have a B/Γ-embedding $B/\Gamma \hookrightarrow X$; then one can construct a G/Γ-embedding $G/\Gamma \hookrightarrow G*_B X$. The new variety $G*_B X$ is a fibre bundle over $G/B \cong \mathbb{P}^1$ with fibre isomorphic to X. Now X is a rational surface, so if it is smooth and complete, we can study X using the theory of smooth projective rational surfaces. In this chapter we study these B/Γ-embeddings. In the first section, we show that any such embedding is obtained from blowing up an embedding where X is a minimal model as a variety, and we review the minimal rational models. In the following section we calculate the actions of B on these minimal models which yield embeddings of B/Γ. In the following chapter, we will see which of the $SL(2,k)/\Gamma$-embeddings are obtained in this way.

Remember that a B/Γ-embedding is a variety X with a regular action of B such that there is an open orbit B-isomorphic to B/Γ. For this chapter, we change the definition of two embeddings being equivalent as follows. The embeddings X_1 and X_2 are equivalent if the are B-isomorphic. This is different from the previous definition in that we do not take into account a base point. The reason for this difference is that now we are interested in the geometry of the orbits in X as a whole whereas in the previous chapters we first studied the embedding locally and then pieced orbits together. For this process one must make sure to keep the same base point for gluing the pieces together. It is not problematic to make this difference, because once an embedding is given, changing the base point amounts to "translating" all the local rings $\mathcal{O}_{\ell}$ of the orbits to $\mathcal{O}_{\ell}s$ for some $s\in B$.

§1. Minimal embeddings: definitions and preliminary results

As usual, B denotes a Borel subgroup of SL(2,k). Let Γ be a finite subgroup of B (so Γ is cyclic).

Given a smooth complete B/Γ-embedding X with fixed point P, The action of B on X induces an action on $\tilde{X}$, the variety obtained by blowing up P in X, giving $\tilde{X}$ the structure of a B/Γ-embedding. We say that X is a minimal B/Γ-embedding if it in not the blow up of another smooth B/Γ-embedding. If X is a minimal model as a variety (that is, if the underlying variety of X is not the blow up of another smooth variety), then clearly X is a minimal embedding. We will now prove the converse.

Proposition 3.1.1. Suppose X is a smooth complete surface on which a connected linear algebraic group H acts regularly. Suppose also that X contains an irreducible curve C with self-intersection $n<0$. Then C is stable by H.

Proof.

Let $s\in H$. Then since H is connected and the action is regular, sC is linearly equivalent to C [6]. Also sC is irreducible, so if $sC\neq C$, then $sC\cap C$ is a finite number of points. Thus the intersection number of C and sC is non-negative; but it must be the self-intersection number of C, which is strictly negative. This is impossible, so $sC=C$ for all $s\in H$.

□

Corollary 3.1.2. Suppose X is a minimal B/Γ-embedding. Then S is a minimal model as a variety; that is, X is a rational minimal model.

Proof.

If X is not a minimal model as a variety, then it contains an irreducible curve C isomorphic to $\mathbb{P}^1$ with self-intersection -1. If we apply the proposition to the case H=B, we see that C is stable by B. So the surface obtained by blowing down C is also a B/Γ-embedding, and X is not a minimal embedding. Also X must be rational because B is rational.

□

We recall the description of the set of minimal models of rational surfaces (see for example [1], [4] or [11]). For any integer $n\geq 0$, define $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. Then $\mathbb{F}_n$ is a ruled surface over $\mathbb{P}^1$. For example, $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1$ is the blow up of $\mathbb{P}^2$ in one point. The set of minimal rational models is given by $\mathbb{P}^2$ and $\mathbb{F}_n$, $n\neq 1$. The $\mathbb{F}_n$ are called Hirzebruch surfaces.

Let us review some elementary properties of the surfaces $\mathbb{F}_n$. As mentioned above, $\mathbb{F}_n$ is a ruled surface over $\mathbb{P}^1$; that is, it is a $\mathbb{P}^1$-fibre bundle over $\mathbb{P}^1$. We restrict to the case $n\geq 1$. Then there is exactly one ruling of $\mathbb{F}_n$, i.e. there is exactly one morphism $\pi_n : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ with fibres isomorphic to $\mathbb{P}^1$. The bundle $\pi_n : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ has a unique section E_n with self-intersection -n, and E_n is the only irreducible curve of $\mathbb{F}_n$ with negative self-intersection. The fibres of π_n are all linearly equivalent, and they are the only irreducible curves with self-intersection 0. So any automorphism of $\mathbb{F}_n$ fixes E_n and permutes the fibres. Now $\mathbb{F}_n - E_n$ is the total space of the vector bundle $\mathcal{O}(n)$ over $\mathbb{P}^1$. All the sections of $\mathcal{O}(n)$ are linearly equivalent (as divisors of $\mathbb{F}_n$) with self-intersection n. Also we have an exact sequence

$$1 \longrightarrow k^* \ltimes H^0(\mathbb{P}^1, \mathcal{O}(n)) \longrightarrow \mathrm{Aut}\, \mathbb{F}_n \xrightarrow{\phi} PGL(2) \longrightarrow 1$$

where ϕ is the restriction to the automorphism of $E_n \cong \mathbb{P}^1$, and k^* acts on $H^o(\mathbb{P}^1,\mathcal{O}(n))$ by multiplication.

We define an action of Aut $\mathbb{F}_n$ on $H^o(\mathbb{P}^1,\mathcal{O}(n))$ as follows. If $\varphi \in \text{Aut } \mathbb{F}_n$ and s is a global section of $\mathcal{O}(n)$, then φs is the section given by $(\varphi s)(x) = \varphi(s(\varphi^{-1}x))$, where $x \in \mathbb{P}^1$ and the action of φ^{-1} on $\mathbb{P}^1$ is given by its action on $E_n \cong \mathbb{P}^1$. Thus $(\varphi s)(\mathbb{P}^1) = \varphi(s(\mathbb{P}^1))$.

Lemma 3.1.3. Let $\varphi \in \text{Aut } \mathbb{F}_n$, $n \geq 1$; then the action of φ on the vector space $H^o(\mathbb{P}^1\mathcal{O}(n))$ given above is an affine transformation.

Proof.

Given $x \in \mathbb{P}^1$ the restriction of φ to the fibre $\varphi^{-1}(\pi_n^{-1}x)$ gives an isomorphism $k \cong \varphi^{-1}(\pi_n^{-1}x) \xrightarrow{\sim} \pi_n^{-1}x \cong k$; this transformation is affine.

Suppose s_1 and s_2 are elements of $H^o(\mathbb{P}^1,\mathcal{O}(n))$; let $s = ts_1 + (1-t)s_2$, $t \in k$. Then for any $x \in \mathbb{P}^1$ we have $(\varphi s)x = \varphi(s(\varphi^{-1}x)) = \varphi(ts_1(\varphi^{-1}x) + (1-t)s_2(\varphi^{-1}x)) = t\varphi(s_1(\varphi^{-1}x)) + (1-t)\varphi(s_2(\varphi^{-1}x)) = t(\varphi s_1)x + (1-t)(\varphi s_2)x$. This proves the lemma.

□

Corollary 3.1.4. For $n \geq 1$, there is a homomorphism

Aut $\mathbb{F}_n \longrightarrow \text{Aff}(H^o(\mathbb{P}^1,\mathcal{O}(n))$ given by $\varphi \longmapsto (s \longrightarrow \varphi s)$.

To describe a B/Γ-embedding with underlying variety X , we must give a homomorphism $B \longrightarrow \text{Aut } X$ such that X has an open orbit B-isomorphic to B/Γ. Two such homomorphisms give rise to equivalent embeddings (remember that we do not fix the base point!) if and only if they are conjugate.

In the following section we will use the information given here to study the possible B/Γ-embeddings into $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_n$, $n \geq 1$.

§2. The minimal B/Γ-embeddings

Proposition 3.2.1. Let Γ be a finite subgroup of B of order c. Then, up to equivalence, the number of B/Γ-embeddings into

(i) $\mathbb{P}^2$ is 3 if c=4 and otherwise 2;

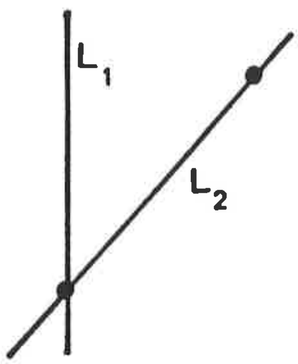
(ii) $\mathbb{P}^1 \times \mathbb{P}^1$ is 2 if c=2 and otherwise 1;

(iii) $\mathbb{F}_n$, $n \geq 1$ is n+4 if c=2(n+1) and otherwise n+3.

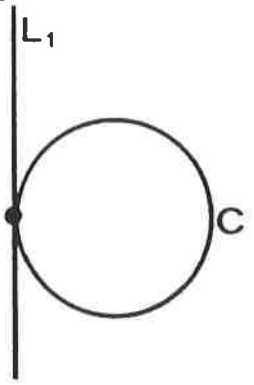
The irreducible components of the complement to the open orbit are always isomorphic to $\mathbb{P}^1$, and they intersect transversely except in one case where c=4 and B/Γ embeds into $\mathbb{P}^2$. In this case there is exactly one fixed point. Also, if c is even, the for one case where c=2(n+1), $n \geq 0$, there is exactly one fixed point. All the other cases have at least two fixed points.
(We include the case $\mathbb{F}_1$ even though it is not minimal.)

To be more precise, we indicate the form of the complement Z to the open orbit in each case. Also to distinguish the embedding where Z has the same form, we indicate how the action of B differs on Z. Let U be the unipotent radical of B and T be a maximal torus. Then B is $T \ltimes U$, and the characters of B are the characters of T. We denote the character group of B by $\{\alpha^n: n \in \mathbb{Z}\}$.

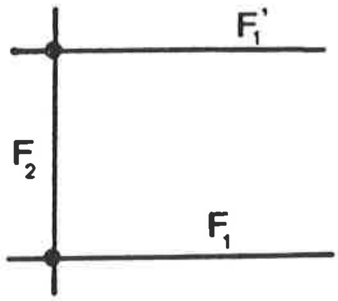
For embeddings into $\mathbb{P}^2$, we find for each Γ, there are two embeddings where $Z=\{L_1 \cup L_2\}$, and L_1 and L_2 are lines in $\mathbb{P}^2$. The unipotent subgroup U acts non-trivially on L_1, and B acts by the character α^{2+c} or α^{2-c} on L_2. (So there are two fixed points except in an embedding for the case c=2, where L_2 is a line of fixed points.)



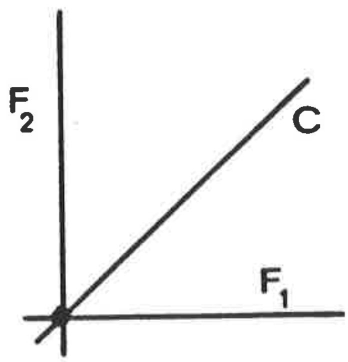
If c=4, we also find an embedding where $Z=\{L_1 \cup C\}$ and C is a smooth conic which is tangent to L_1 at the unique fixed point.



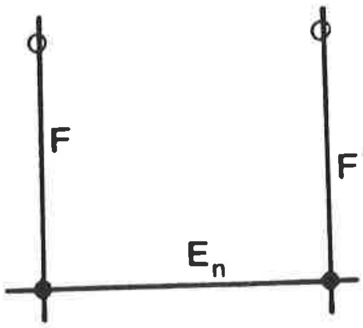
For embeddings into $\mathbb{P}^1 \times \mathbb{P}^1$, Z is always the union of three curves. Let $p_i:\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, i=1,2 be the two projections. Then for each r there is an embedding where $Z=\{F_1 \cup F_1' \cup F_2\}$ and F_1, F_1' are fibres of p_1 and F_2 is a fibre of p_2. There are two fixed points.



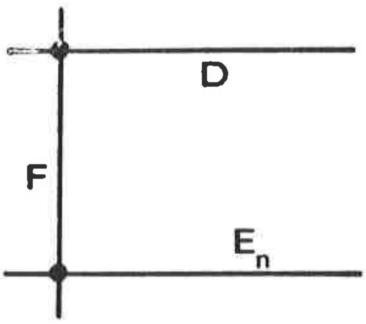
Also, if c=2, we find another embedding into $\mathbb{P}^1 \times \mathbb{P}^1$ where $Z=\{F_1 \cup F_2 \cup C\}$ and C is a section of p_1 and p_2 which intersects F_1 and F_2 transversely in the unique fixed point.



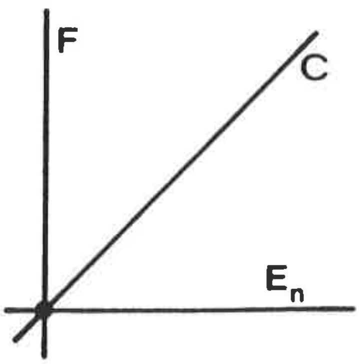
For embeddings into $\mathbb{F}_n$. $n \geq 1$, again Z is is always the union of three curves. Let $\pi_n:\mathbb{F}_n \longrightarrow \mathbb{P}^1$ be the unique ruling of $\mathbb{F}_n$, and let E_n be the irreducible curve of $\mathbb{F}_n$ with self-intesection $-n$. For each Γ we find $n+1$ cases where $Z=\{E_n \cup F \cup F'\}$ and F and F' are fibres of π_n. The torus T acts on F by the character α^{cp+2} and on F' by the character $\alpha^{-c(n-p)+2}$, $p=0,\ldots,n$. (There are either 3 or 4 fixed points, or, if T acts trivially on F', then F' is a curve of fixed points.)



There are also two other embeddings in $\mathbb{F}_n$ for each Γ where $Z=\{F \cup E_n \cup D\}$ and F is a fibre as before and D is a section of π_n which does not intersect E_n. The group B acts on F by the character $\alpha^{2n\pm c}$. (There are two fixed points except in one of the embeddings in the case where $c=2n$, in which case F consists of fixed points.)



Also if $c=2(n+1)$, there is one more embedding where $Z = \{E_n \cup F \cup C\}$ and C is a section which intersects E_n and F transversely in the unique fixed point.



This embedding is obtained as follows. Consider the embedding into $\mathbb{F}_{n+1}$ of the previous type where the fibre F consists of fixed points. Blow up a point of F which is not on E_{n+1} or D and contract the strict transform of F. This gives the required embedding into $\mathbb{F}_n$.

<u>Proof of the Proposition.</u>

Recall that to give an embedding of B/Γ into a variety X, we must find a homomorphism $\varphi: B \longrightarrow \mathrm{Aut}\, X$ such that under the induced action of B on X, there is an open orbit isomorphic to B/Γ. Two such embeddings are equivalent (under the equivalence considered in this chapter) if and only if the homomorphisms are conjugate.

Set $B = \{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \epsilon k^* \text{ and } \beta \epsilon k\}$, $U = \{\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \epsilon k\}$ and $T = \{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \epsilon k^*\}$.

We consider separately the embeddings into $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_n$, $n \geq 1$.

<u>Embeddings into $\mathbb{P}^2$:</u>

If B acts on $\mathbb{P}^2$, it has a fixed point d since $\mathbb{P}^2$ is complete and B is solvable. Also B acts on the linear system S={lines of $\mathbb{P}^2$ passing through d}. Since we have $S \cong \mathbb{P}^1$, B fixes one such line, which we call L.

<u>CASE 1. U acts trivially on L.</u>

Then there is another point $d' \epsilon L$ fixed by B. By choosing an appropriate basis, we can assume that for $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \epsilon$ U we have

$$\varphi\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \epsilon \, PGL(3)$$

and $\varphi(B)$ is upper triangular. By a change of basis we can also assume that $\varphi(T)$ is diagonal. Then for φ to be a homomorphism, it is necessary that

$$\varphi\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{bmatrix} \alpha^m & 0 & 0 \\ 0 & \alpha & \beta \\ \alpha & 0 & \alpha^{-1} \end{bmatrix} \epsilon \, PGL(3), \quad m \epsilon \mathbb{Z}.$$

For $m=-1\pm c$, this gives an embedding of B/Γ with $|\Gamma|=c$. In this basis, $L = \{(z_0:z_1:0) \mid z_i \in k\}$, and B acts on L by the character $\alpha^{2\pm c}$. There is another fixed line $\{(0:z_1:z_2) \mid z_i \in k\}$ on which U acts non-trivially. This gives two B/Γ-embeddings mentioned earlier for $\mathbb{P}^2$.

CASE 2. U acts non-trivially on L.

(i) U acts trivially on the linear system S.

Then B fixes another line L′ passing through d, and the complement to the open orbit is $Z= L \cup L'$. So U acts trivially on L′; indeed, let D be a line of $\mathbb{P}^2$ not passing through d and let $u \in U$, $u \neq e$; then $uD \cap D$ is a point fixed by u; it therefore must belong to Z, but it is not in L; thus it is in L′. So by exchanging L and L′, we are in Case 1.

(ii) U acts non-trivially on the linear system S.

Then fix $u \in U$, $u \neq e$. We can choose a basis such that $\gamma(u)$ is in Jordan normal form $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Now by a change of basis we can assume

$$\gamma\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 2\beta & \beta^2 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \in PGL(3)$$

and $\gamma(T)$ is diagonal. There is just one possibility:

$$\gamma\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{bmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ 0 & 1 & \alpha^{-1}\beta \\ 0 & 0 & \alpha^{-2} \end{bmatrix} \in PGL(3).$$

(So γ is obtained from the irreducible representation of SL(2) of dimension 3.) This homomorphism gives rise to a B/Γ-embedding for $c=4$. The complement to the open orbit consists of two components: $L = \{(z_0:z_1:0)\}$ and $C = \{(z_0:z_1:z_2) \mid z_0z_2-z_1^2 = 0\}$, and there is exactly one fixed point: (1:0:0).

Embeddings into $\mathbb{P}^1 \times \mathbb{P}^1$:

The two projections $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ give the two different rulings of

$\mathbb{P}^1 \times \mathbb{P}^1$. Any automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ either fixed the two rulings or exchanges them. In other words, $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) = (PGL(2) \times PGL(2)) \rtimes \mathbb{Z}/2\mathbb{Z}$. Since B is connected, the image of $\varphi(B) \hookrightarrow \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is connected; thus we consider homomorphisms $\varphi: B \longrightarrow PGL(2) \times PGL(2)$. Up to conjugation, the only homomorphisms of B to PGL(2) are

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longrightarrow \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \in PGL(2)$$

or $$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longrightarrow \begin{bmatrix} \alpha^m & 0 \\ 0 & 1 \end{bmatrix} \in PGL(2), \quad m = 0,1,2,\ldots.$$

To obtain an embedding, U cannot act trivially on $\mathbb{P}^1 \times \mathbb{P}^1$. So the possib ilities are

$$\varphi\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \{ \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} \alpha^m & 0 \\ 0 & 1 \end{bmatrix} \} \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1),\ m=1,2,3,\ldots$$

or $$\varphi\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \{ \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \} \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1).$$

In the first case, we get an embedding of B/Γ with c=m; the second induces a B/Γ-embedding with c=2, and the complement to the open orbit consists of three curves isomorphic to $\mathbb{P}^1$ all intersecting transversely in the unique fixed point. So for c=2, we find two embeddings, and for c≠2 we find one.

<u>Embeddings into $\mathbb{F}_n$, n≥1:</u>

Remember from section 1 that we can consider $\mathbb{F}_n$ as the union of E_n and the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$. Suppose we have a homomorphism $\varphi: B \longrightarrow \mathrm{Aut}\ \mathbb{F}_n$ which gives rise to a B/Γ-embedding. Since $\mathrm{Aut}\ \mathbb{F}_n$ fixes E_n, we know that B fixes E_n. We consider three cases.

<u>CASE 1. U acts trivially on E_n.</u>

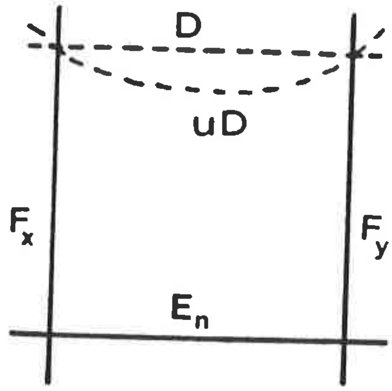
We will find n+1 inequivalent embeddings of this type for each Γ.

In this case, consider the action of T on E_n. It cannot act trivially (because then each B-orbit would be contained in a fibre of $\pi_n: \mathbb{F}_n \longrightarrow \mathbb{P}^1$) and has therefore exactly two fixed points, x and y. By possibly exchanging x and y, we can assume that T acts by a character α^m, m>0 on $E_n \cong \mathbb{P}^1$ (i.e. for $z \in E_n - \{x,y\}$, we choose $x = \lim_{t \to 0} tz$ and $y = \lim_{t \to \infty} tz$, $t \in T$).

The fibres F_x and F_y of x and y, respectively, are stable by B. Let Z be the complement of the open orbit in $\mathbb{F}_n$. Then we have $E_n \cup F_x \cup F_y \subset Z$. Since we know that $\mathbb{F}_n - \{E_n \cup F_x \cup F_y\} \cong k \times k^* \cong B/\Gamma$, and since $k \times k^*$ contains no proper open subvariety isomorphic to itself, we must have $Z = E_n \cup F_x \cup F_y$.

Now by Corollary 3.1.4, we have $T \hookrightarrow B \longrightarrow \mathrm{Aut}\, \mathbb{F}_n \longrightarrow \mathrm{Aff}(H^0(\mathbb{P}^1, \mathcal{O}(n))$. Since T is reductive, T must fix a section D of $\mathcal{O}(n)$.

We also have that U acts on the vector space $H^0(\mathbb{P}^1, \mathcal{O}(n))$. Consider the orbit UD. First note that $UD \cong k$ (we could not have UD = D, because then D would be in the complement of the open orbit). Now let $u \in U$, $u \neq e$; then I claim that $uD \cap D \subset \{x', y'\}$, where $x' = F_x \cap D$ and $y' = F_y \cap D$. To see this, note that since U acts trivially on E_n, it stabilized the fibres of π_n. Thus if z belongs to $uD \cap D$, then u belongs to the isotropy group of z, and therefore z must be in Z. The intersection number $uD \cdot D$ is n; so $UD \subset D \cup \bigsqcup_{p=0}^{n} A_p$, where A_p is the set of sections D' of $\mathcal{O}(n)$ such that $D \cap D' = px' + (n-p)y'$ counted with multiplicity. Now $D \cup A_p$ is isomorphic to k, p=0,...,n; so $UD = D \cup A_p$ for some p=0,...,n. We call p the <u>index</u> of the embedding.



<u>Lemma 3.2.2.</u> Up to equivalence, there is at most one B/Γ-embedding into $\mathbb{F}_n$ of a given index p, with $p=0,\ldots,n$.

<u>Proof.</u>

Suppose we have two B/Γ-embeddings into $\mathbb{F}_n$ with the same index p. Fix $u \in U$, $u \neq e$. For the first (resp. second) action denote by x,y (resp. $\tilde{x},\tilde{y}$) the fixed points in E_n and D (resp. $\tilde{D}$) the section fixed by T. Set $D_u := uD$ (resp. $\tilde{D}_u = u\tilde{D}$).

By conjugating by an automorphism of $\mathbb{F}_n$ which permutes the fibres, we can assume $x=\tilde{x}$ and $y=\tilde{y}$. Then by conjugating by an automorphism which fixes the fibres and translates the sections, we can assume $D=\tilde{D}$. Finally, since the two embeddings have the same index, by conjugating by an automorphism that fixes the fibres and which is a homothety centered at D, we can assume $D_u=\tilde{D}_u$.

Now I claim that for a fixed Γ, there is at most one possible action of B on $\mathbb{F}_n$ which induces a B/Γ-embedding with the quadruple $\{x,y,D,D_u\}$. Indeed U acts by translation on each of the fibres of $\mathcal{O}(n)$; so D and D_u determine how U must act. Now check the action of T on D, which is the same as its action on E_n. Choose $z \in D$ in the open orbit. The order of the isotropy group B_z is c, the order of Γ, and $B_z \subset T$. So T acts on D by a character $\alpha^{\pm c}$. Since we chose x and y such that the action of T on E_n is given by a positive character, we must have that T acts on D by the character α^c. Now let v be an element of the open orbit and $t \in T$. Choose $u \in U$ such that $(t^{-1}ut)v = v' \in D$. Then $tv = u^{-1}tv'$. So this fixes the action of T on the open orbit, which is dense in $\mathbb{F}_n$. So the claim is true, and this finishes the proof of the lemma.

□

By this lemma, we have at most n+1 inequivalent embeddings for each G/Γ of this type. Now we must show that these actually exist.

<u>Lemma 3.2.3.</u> Let n be a positive integer and p be an integer such that $0 \leq p \leq n$. Then for each finite $\Gamma \subset B$, there exists a B/Γ-embedding into $\mathbb{F}_n$ with index p.

Proof.

Remember that if one contracts the section E_n of $\mathbb{F}_n$, one obtains a surface X_n (nonsingular if and only if n=1) contained in $\mathbb{P}^{n+1}$. In fact X_n is the closure of the affine cone over the n-tuple embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^n$. Giving an embedding of B/Γ is equivalent to giving an embedding of B/Γ into X_n which fixes the "center" of the cone (if n>1, this condition is always satisfied, because this point is singular).

For each p with $0 \le p \le n$, we will exhibit an action of B on X_n which induces a B/Γ-embedding with index p. To do this we give a linear action of B on k^{n+2} which induces an action of B on $\mathbb{P}^{n+1}$ stabilizing X_n and its "center."

B acts on k^2 in the usual way:

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \alpha s+\beta t \\ \alpha^{-1} t \end{pmatrix}.$$

Also for $i \in \mathbb{Z}$, we denote by (k,α^i) the variety k with the action of B by the character α^i:

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} z = \alpha^i z.$$

Consider the B-module

$$k^2 \oplus (k,\alpha^{cp+1}) \oplus \bigoplus_{\substack{j=0 \\ j \neq p}}^{n} (k,\alpha^{cj}), \quad p=0,\ldots,n.$$

We have $B \longrightarrow PGL(n+2) = \mathrm{Aut}\, \mathbb{P}^{n+1}$ by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \begin{bmatrix} \alpha^{cp+2} & \alpha^{cp+1}\beta & 0 & \cdots & & & 0 \\ & \alpha^{cp} & & & & & \\ & & 1 & & & 0 & \\ & & & \alpha^{c} & & & \\ & & 0 & & \ddots & & \\ & & & & & \widehat{\alpha^{cp}} & \\ & & & & & & \ddots \\ & & & & & & \alpha^{cn} \end{bmatrix}$$

We change the basis so that the image of $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$ is

$$\begin{bmatrix} \alpha^{cp+2} & 0 & \cdots & 0 & \alpha^{cp+1}\beta & 0 & \cdots & 0 \\ & 1 & & & & & & \\ & & \alpha^{c} & & & \mathcal{O} & & \\ & & & \ddots & & & & \\ & \mathcal{O} & & & \alpha^{cp} & & & \\ & & & & & \ddots & & \\ & & & & & & & \alpha^{cn} \end{bmatrix}.$$

Let $X_n = \{(z_0:s^n:s^{n-1}t:\ldots:t^n) \mid z_0,s,t\in k\} \subset \mathbb{P}^{n+1}$. Clearly X_n and the center of the cone $(1:0:\ldots:0)$ are fixed by this action. In X_n the two "fibres" $F_x = \{(z_0:z_1:0:\ldots:0)\}$ and $F_y = \{(z_0:0:\ldots:0:z_n)\}$ are stable. It is easy to check that the isotropy group of $(0:1:\ldots:1)$ is the finite subgroup of T of order c. So this induces an embedding of B/Γ into X_n which by blowing up the center gives a B/Γ-embedding into $\mathbb{F}_n$ where U acts trivially on E_n.

Let $D = \{(0:s^n:s^{n-1}t:\ldots:t^n)\} \subset X_n$. Then D is a "section" stable by T. Fix $u=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U$. Then $uD = \{(s^{n-p}t^p:s^n:s^{n-1}t:\ldots:t^n)\} \subset X_n$. We check the multiplicity of the intersection of D and uD at $x'=(0:1:0:\ldots:0)$. The local ring of x' in X_n is $k[z_0,t]_{(t,z_0)}$, and the local equation of D (resp. uD) is $z_0=0$ (resp. $z_0= t^p$); thus this multiplicity is p, and the index of the embedding is p. This finishes the proof of the lemma.

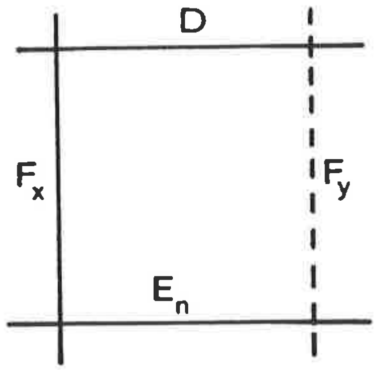
□

<u>Remark.</u> By checking the action of the torus on the fibres of the actions found in Lemma 3.2.3, we find for this case the result mentioned in the remarks following the statement of the proposition.

CASE 2. U acts non-trivially on E_n and B fixes a section D of $\mathcal{O}(n)$.

We will find two embeddings of this type for each Γ.

In this case, U has one fixed point x on E_n. Then T must also fix x, and it also fixes another point $y \in E_n$. As before, we call Z the complement to the open orbit. Then we have $Z = E_n \cup D \cup F_x$, where F_x is the fibre of π_n containing x. Now look at the action of T on F_y, the fibre of y. Choose $z \in F_y$ in the open orbit. Then the order of the isotropy group B_z is c, the order of Γ, and $B_z \subset T$. So T acts on F_y by the character $\alpha^{\pm c}$. For each such embedding, call this character the signature of the embedding.



Lemma 3.2.4. Up to equivalence, there is at most one B/Γ-embedding into $\mathbb{F}_n$ with a given signature $\sigma = \alpha^{\pm c}$.

Proof.

Suppose we had two actions of B on $\mathbb{F}_n$ which yield two such embeddings. For the first (resp. second) action, let ψ (resp. $\tilde{\psi}$) $: B \times E_n \longrightarrow E_n$ be the induced action on E_n and D (resp. $\tilde{D}$) be the section of $\mathcal{O}(n)$ fixed by B.

Up to conjugacy there is only one action of B on $E_n \cong \mathbb{P}^1$ for which U acts non-trivially. So we can assume $\psi = \tilde{\psi}$. By conjugating by an automorphism of $\mathbb{F}_n$ which fixes the fibres and translates the sections, we can assume $D = \tilde{D}$.

Now I claim there is at most one action of B on $\mathbb{F}_n$ which yields a B/Γ-embedding with the triple $\{\psi, D, \sigma\}$. To see this, consider first the action of U on $\mathbb{F}_n$. Now x is the fixed point of E_n, and F_x is its fibre. Let S be the set of sections of $\mathcal{O}(n)$ which are not D and intersect D with multiplicity

n at the fixed point $x'=F_x \cap D$. This set is isomorphic to k^* and is stable by B, so U acts trivially on S. Since the action of U on $D' \in S$ is identical to its action on E_n, the action of U on $\mathbb{F}_n$ is determined by γ and D. As for the action of T, remember that T stabilizes the set S. The action on this set is equivalent to its action on F_y, the fibre of the point of E_n fixed by T and not fixed by U. This action is given by σ. So $\{\gamma, D, \sigma\}$ determines the action of T on $\mathbb{F}_n$. This proves the claim.

□

From this lemma, we see that for each Γ, there is at most two B/Γ-embeddings of this type. Now we must show that these embeddings actually exist.

Lemma 3.2.5. Let Γ be a finite subgroup of B of order c and σ be $\alpha^{\pm c}$. Then there exists a B/Γ-embedding into $\mathbb{F}_n$ with signature σ.

Proof.

We use the same notation as in Lemma 3.2.3. Consider the B-module

$$(k, \alpha^{-n\pm c}) \oplus S^n(k^2)$$

where $S^n(k^2)$ is the vector space of homogeneous polynomials of degree n over k with two variables, and the action of B on $S^n(k^2)$ is induced from the action given in Lemma 3.2.3 on k^2. We have $B \longrightarrow PGL(n+2)$ by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \begin{bmatrix} \alpha^{-n\pm c} & 0 & \cdots & 0 \\ 0 & & & \\ \cdot & & \rho_n\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} & \\ \cdot & & & \\ \cdot & & & \\ 0 & & & \end{bmatrix}$$

where ρ_n is the $(n+1)$-dimensional irreducible matrix representation of $SL(2,k)$ corresponding to the basis $\{\binom{n}{i} x^i y^{n-i}\}_{i=0,\ldots,n}$ of $S^n(k^2)$.

As in Lemma 3.2.3, let $X_n = \{(z_0 : s^n : s^{n-1}t : \ldots : t^n) \mid z_0, s, t \in k\} \subset \mathbb{P}^{n+1}$. Then X_n and its center $(1:0:\ldots:0)$ are fixed by the action above. In X_n the "section" $\{(0:s^n:\ldots:t^n)\}$ and the "fibre" $\{(z_0:z_1:0:\ldots:0)\}$ are stable. The isotropy group of (1:0:...:0:1) is the finite subgroup of T of order c. So this action gives an embedding of B/Γ into X_n which by blowing up the center gives an embedding into $\mathbb{F}_n$ where U acts non-trivially on E_n and B fixes a section.

The "fibre" $\{(z_0:0:\ldots:0:z_n)\}$ is stable by T and not by U. Also T acts on this fibre by the character $\alpha^{\pm c}$, so the signature of the embedding is $\alpha^{\pm c}$. This proves the lemma.

□

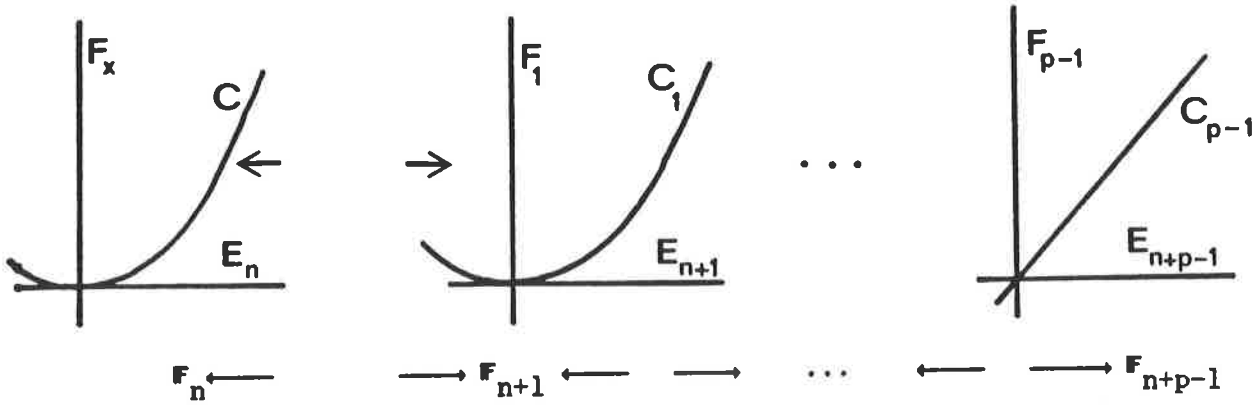
<u>Remark.</u> The group B acts on the fixed fibre of the B/Γ-embedding with signature $\alpha^{\pm c}$ by the character $\alpha^{2n\mp c}$. In particular, for each n, there is exactly on embedding of this type for c=2n where B acts trivially on the fixed fibre. We will use this remark for the following case.

<u>CASE 3. U acts non-trivially on E_n and B does not fix any section of $\mathcal{O}(n)$.</u>

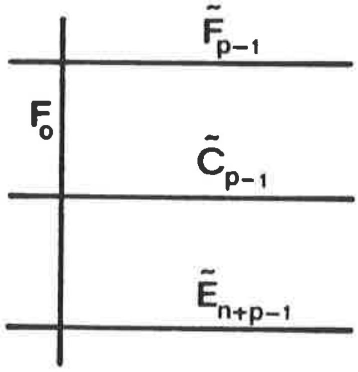
For each n, we find one such case where c=2(n+1).

As in the previous case, B fixes one element $x \in E_n$. So Z, the complement to the open orbit contains E_n and F_x, the fibre of x. Now $\mathbb{F}_n - \{E_n \cup F_x\}$ is isomorphic to $k \times k$; so Z must have another component. Suppose $z \in Z - \{E_n \cup F_x\}$; then $C = \overline{Bz}$ is contained in Z. Clearly C is a section of $\pi_n: \mathbb{F}_n \longrightarrow \mathbb{P}^1$, and by hypothesis it in not a section of $\mathcal{O}(n)$; thus it is a section of π_n which intersects E_n at the point x. We have $Z = E_n \cup F_x \cup C$, since $\mathbb{F}_n - \{E_n \cup F_x \cup C\} \cong k \times k^*$.

The intersection number $C \cdot F_x$ is 1 since C is a section and F_x is a fibre, and $C \cdot E_n = p$ is strictly positive. Now blow up x and then contract the strict transform of F_x; we obtain an embedding into $\mathbb{F}_{n+1}$. Let C_1 be the strict transform of C in $\mathbb{F}_{n+1}$; then the intersection number $C_1 \cdot E_{n+1}$ is p-1. Also, this new embedding has at least two fixed points: one on E_{n+1} and the other the image of the strict transform of F_x in $\mathbb{F}_{n+1}$. By doing this process p-1 times, we get an embedding into $\mathbb{F}_{n+p-1}$ with $Z = E_{n+p-1} \cup F_{p-1} \cup C_{p-1}$, where F_{p-1} is a fibre and C_p is a section of $\pi_{n+p-1}: \mathbb{F}_{n+p-1} \longrightarrow \mathbb{P}^1$ with $C_{p-1} \cdot E_{n+p-1} = 1$.



Now blow up x once more and blow down the strict transform of F_{p-1}. We obtain an embedding where the complement of the open orbit has the form



Now F_0 has three fixed points, so B acts trivially on F_0. Blow down $\tilde{F}_{p-1}$; we obtain an embedding into $\mathbb{F}_{n+p}$ as in Case 2, where B acts trivially on the fixed fibre. As we have seen in the remark of Case 2, this happens in exactly one case with $c=2(n+p)$. Thus for $c=2(n+p)$ we find there is at most one embedding into $\mathbb{F}_{n+p-1}$ of Case 3 with $C_{p-1} \cdot E_{n+p-1} = 1$. Now we will show that $p=1$. This is done as follows. Suppose $p>1$. Then by the process of blowing up and down described above, we obtain an embedding into $\mathbb{F}_{n+p-1}$ with at least two fixed points. By exhibiting the embedding into $\mathbb{F}_{n+p-1}$ with $C_{p-1} \cdot E_{n+p-1} = 1$, we will see that it has a unique fixed point. Thus the Case 3 is finished by the following lemma.

Lemma 3.2.6. Let Γ be a finite subgroup of B of order c. If $c=2(n+1)$, there exists a B/Γ-embedding into $\mathbb{F}_n$ of Case 3 with $C\cdot E_n=1$, and this embedding has a unique fixed point.

Proof.

We use the notation of Lemmas 3.2.3 and 3.2.5.

Consider the B-module $S^{n+1}(k^2)$. We have $B \longrightarrow PGL(n+2)$ by

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \left[\rho_{n+1}\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}\right]$$

where ρ_{n+1} is the (n+2)-dimensional irreducible representation of $SL(2,k)$.

Consider the closure of the orbit of $x^{n+1}+y^{n+1}$ by B using the basis $\{\binom{n+1}{i}x^iy^{n+1-i}\}_{i=0,\dots,n+1}$. This is exactly $X_n = \{(z_0:s^n:s^{n-1}t:\dots:t^n) \mid z_0,s,t\epsilon k\}$. The center $(1:0:\dots:0)$ is fixed by this action. The two stable curves in X_n are the "fibre" $\{(z_0:z_1:0:\dots:0)\}$ and $\{(s^{n+1}:s^nt:\dots:t^{n+1})\}$, the image of the (n+1)-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^{n+1}$. It is easy to see that the isotropy group of $(1:0:\dots:0:1)$ is the finite subgroup of T of order c; so this action gives a B/Γ-embedding into X_n which induces an embedding into $\mathbb{F}_n$. Since the only fixed point on X_n is the center and there is only one fixed "fibre", we have exactly one fixed point for the action on $\mathbb{F}_n$. It is easily checked that the intersection number of E_n with the other stable section in $\mathbb{F}_n$ is 1.

□

This finishes Case 3. Thus we know all the embeddings into $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and $\mathbb{F}_n$, $n\geq 1$. The remarks at the end of the proposition are easily verified by checking each embedding. This finishes the proof of the proposition.

□

Remarks.

(1) Note that - as to be expected - all the embedding into $\mathbb{F}_1$ are obtained by blowing up the embeddings into $\mathbb{P}^2$ at fixed points.

(2) The "exceptional embeddings", that is, those with only one fixed point, are of special interest, because this phenomenon does not occur for smooth complete embeddings of tori.

CHAPTER IV: THE RELATIONSHIP BETWEEN G/Γ-EMBEDDINGS AND B/Γ-EMBEDDINGS

As in the previous chapters, G denotes the group SL(2,k) and B denotes a fixed Borel subgroup of G. Let Γ be a finite (cyclic) subgroup of B.

In this chapter, we consider the classification of normal G/Γ-embeddings given in Chapter I and deduce a similar classification for the normal B/Γ-embeddings. (It is possible to calculate directly this classification using methods similar to those of Chapter I, but since we already did the work for SL(2), it is simpler to use what we already know.) Then we study the smooth B/Γ-embeddings and calculate how the minimal embeddings found in Chapter III fit into our classification. This gives us information about many of the smooth G/Γ-embeddings. For the cases of $\Gamma=\{e\}$ and $\{\pm e\}$, this will be discussed further in section 7.

§ 1. Classification of normal B/Γ-embeddings from G/Γ-embeddings

The aim of this section is to describe a relation between B/Γ-embeddings and G/Γ-embeddings.

Given an embedding X of B/Γ, we construct an embedding of G/Γ as follows. Consider the action of B on GxX by $b\cdot(s,x)=(sb^{-1},bx)$ for $b\in B$, $s\in G$ and $x\in X$. We denote the quotient of GxX by this action as $G*_B X$. This new variety has a natural action of G, and the open equivariant immersion of B/Γ into X induces an open equivariant immersion of G/Γ into $G*_B X$. Thus, $G*_B X$ is a G/Γ-embedding.

There is a natural morphism $\pi:G*_B X \longrightarrow G/B$ induced by the projection of GxB onto G. In fact $G*_B X$ is a locally trivial fibre bundle over G/B with fibre B-isomorphic to X; this is because the fibre bundle $G/\Gamma \longrightarrow G/B$ is locally trivial.

Proposition 4.1.1. Let ϕ be the map from the set of B/Γ-embeddings to the set of G/Γ-embeddings given by $\phi(X)=G*_BX$. Then

(i) ϕ is injective;

(ii) X is normal (resp. smooth) if and only if $\phi(X)$ is normal (resp. smooth).

Proof.

(i) Suppose $G*_BX_1= G*_BX_2$ (as G/Γ-embeddings). The inclusion of G/Γ into $G*_BX_i$ induces an inclusion of B/Γ into $G*_BX_i$, i=1,2. Now X_i is the closure of B/Γ in $G*_BX_i$, i=1,2, so $X_1=X_2$. (Note that it is essential that we take into consideration the base point for this argument.)

(ii) This follows directly from the facts that $\pi:G*_BX \longrightarrow G/B$ is locally trivial and that G/B is smooth.

□

Now we would like to find the image of ϕ. We are really only interested in the normal embeddings, so now let us restrict ϕ to the normal embeddings of B/Γ.

Proposition 4.1.2. Let X' be a normal embedding of G/Γ. Then X' is in the image of ϕ if and only if no G-orbit in X' is completely contained in the closure of B/Γ in X'.

To prove this proposition we need the following lemma:

Lemma 4.1.3. Let X' be a normal G/Γ-embedding. Then the closure of B/Γ in X' intersects every orbit of X'.

Proof of Lemma.

To prove the lemma we show that $G\cdot\overline{B/\Gamma} = X'$. Clearly $G\cdot\overline{B/\Gamma}$ contains the open orbit of X', so we need only show that it is closed. Indeed the image of the morphism

$$\psi:\ G*_B\overline{B/\Gamma} \longrightarrow X'$$
$$(s,x) \longmapsto sx$$

is $G\cdot\overline{B/\Gamma}$; it therefore suffices to prove that ψ is proper. The morphism ψ can be decomposed as follows:

$$G*_B\overline{B/\Gamma} \xrightarrow{\psi'} G/B \times X' \xrightarrow{p_2} X'$$
$$(s,x) \longmapsto (sB/B,sx) \longmapsto sx.$$

The morphism p_2 is proper since G/B is complete, and ψ' is in fact a closed immersion. So ψ is proper.

□

Proof of Proposition 4.1.2.

Let X be the closure of B/Γ in X'. Then X' is in the image of ϕ if and only if $X' = \phi(X)$. Certainly, if this is the case, no orbit in X' is entirely contained in X.

On the other hand, suppose that no orbit of X' is contained in X. Let $\psi: G*_B X \longrightarrow X'$ be the map defined by $\psi(s,x) = sx$, $s\in G$ and $x\in X$. Then ψ is birational, and by the lemma, it is surjective. We will show that ψ is an isomorphism. By Zariski's Main Theorem it suffices to prove that the fibres of ψ are finite (since X' is normal).

Let $z\in X'$; now $\dim(\psi^{-1}(z)) = \dim(Gz \cap X) + \dim G_z - \dim B < \dim Gz + \dim G_z - \dim B$, since $Gz \cap X \neq Gz$ by hypothesis. (G_z denotes the isotropy subgroup of z.) So $\dim(\psi^{-1}(z)) \leq \dim G - \dim B -1 = 0$, since B is of codimension one in G. Therefore $\dim(\psi^{-1}(z)) = 0$; thus ψ is an isomorphism, and the proposition is proven.

□

This proposition is very useful, because, given the information from Proposition 1.3.1 it is very easy to tell if an orbit is in the closure of B/Γ. The reason is that B/Γ, being of codimension one, is an element of ${}^B\mathcal{D}(G/\Gamma)$; i.e. it is a B-stable irreducible divisor of G/Γ. So a locality $\ell\in L_1^n(G/\Gamma)$ is the locality of an orbit contained in $\overline{B/\Gamma}$ if and only if $B/\Gamma \in \mathcal{D}_\ell$. We check the list of $\mathcal{D}_\ell$'s for $\ell\in L_1^n(G/\Gamma)$, and we find

Corollary 4.1.4. Suppose X' is a normal embedding of G/Γ; then X' is in the image of ϕ if and only if each of its localities are of the following types:

Type A_α with $D_j=B/\Gamma$ for some j, $j=1,\dots,\alpha$;

Type B_+ with $D_1 \neq B/\Gamma$;

Type B_- with $D_1=B/\Gamma$;

Type AB;

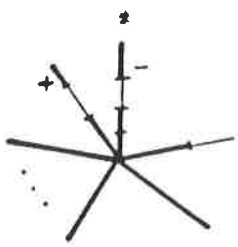
or Type C.

(Note that X' cannot have a fixed point, because it would be entirely in the closure of B/Γ. So we find no localities of type B_o in the list.)

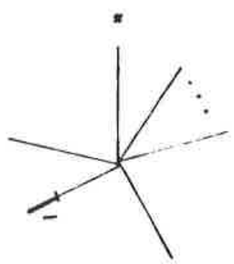
Thus we obtain a classification for the localities of orbits of normal B/Γ-embeddings.

§2. The diagrams of normal B/Γ-embeddings

Given a normal B/Γ-embedding X we will associate to it a diagram similar to the diagram of $G*_BX$. We do this as follows. First draw the diagram of $G*_BX$. Next label $B/\Gamma \in P^1/\Gamma$ by "*". By Corollary 4.1.4, we know which diagrams ar possible. For example, for $\Gamma=\{e\}$



is allowed, but



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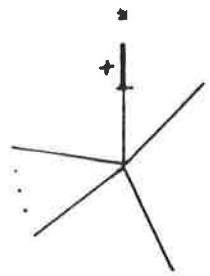

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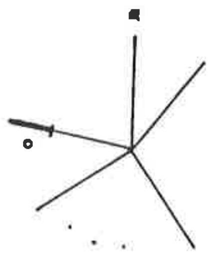

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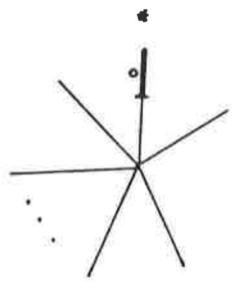

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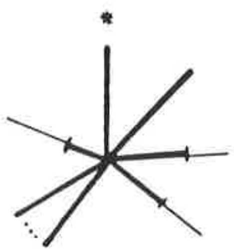
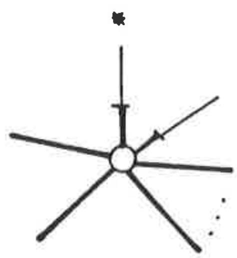


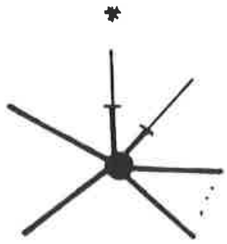
are not.

Now there is no ambiguity between the localities of type B_+ and B_- (and there are none of type B_o), so we can omit the signs "+" and "-" in the diagram. There is one more change we will make. We replace the "center" of the diagram by a small circle, which we darken if X contains a subvariety with locality v(,-1) (i.e. if it has an infinite number of type B_+ orbits). For example



has 4 orbits

whereas



has an infinite number of orbits.

§3. The relationship between local rings of G/Γ-embeddings and local rings of B/Γ-embeddings

Let B^- be a Borel subgroup of G opposed to B, and let U^- be its unipotent radical. Then

$$B^-B = U^-B \cong U^- \times B \xrightarrow{p_2} B$$

induces the inclusion $k[B] \hookrightarrow k[B^-B]$. This in turn induces an inclusion $k(B) \hookrightarrow k(B^-B)=k(G)$; so we consider $k(B) \subset k(G)$.

Suppose Y is an orbit of a B-embedding X with local ring $\mathcal{O}_\ell$. Denote the local ring of $Y' = G*_B Y$ in $X' = G*_B X$ by $\mathcal{O}_{\ell'}$. Then $\mathcal{O}_\ell = k(B) \cap \mathcal{O}_{\ell'}$. To see this, note that $\pi: G*_B X \longrightarrow G/B$ is locally trivial, and $\mathcal{O}_\ell$ is the local ring of the intersection of a fibre with Y' in that fibre.

In the diagram of a normal B/Γ-embedding, the set of "points" represents $\mathcal{V}(B/\Gamma)$ = {discrete normalized B-stable valuations of $k(B)$ over k}; that is, there is a one-to-one correspondence between $\mathcal{V}(G/\Gamma)$ and $\mathcal{V}(B/\Gamma)$ given by $\mathcal{O}_{v'} \longleftrightarrow \mathcal{O}_{v'} \cap k(B)$. For any locality $\ell' \in \mathcal{L}_1^n(G/\Gamma)$ and $v' \in \mathcal{V}(G/\Gamma)$, we have $v' \in \mathcal{F}_{\ell'}$ if and only if $\mathcal{O}_{v'} \cap k(B) = \mathcal{O}_v$ dominates $\mathcal{O}_{\ell'} \cap k(B) = \mathcal{O}_\ell$. This means that the diagram of a normal B/Γ-embedding represents the embedding in much the same way that we have for G/Γ-embeddings. We denote by $\mathcal{L}_1^n(B/\Gamma)$ the set of localities of non-open orbits of normal B/Γ-embeddings. For $\ell \in \mathcal{L}_1^n(B/\Gamma)$, we can define $\mathcal{V}_\ell$ and $\mathcal{F}_\ell$ as for G/Γ-embeddings. Then the darkened part of the diagram for ℓ is $\mathcal{F}_\ell$, and also $v \in \mathcal{V}_\ell$ if and only if the corresponding v' of $\mathcal{V}(G/\Gamma)$ is in $\mathcal{V}_{\ell'}$, where $\mathcal{O}_\ell = \mathcal{O}_{\ell'} \cap k(B)$.

For the B/Γ-embeddings, the orbits with localities of type C are one-dimensional, and those with localities of type B_+, B_- or A_α are fixed points.

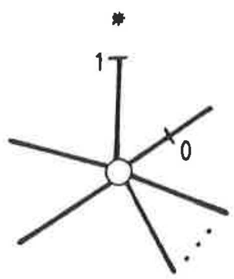
<u>Example.</u>

Let $B = \{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in k^*, \beta \in k\}$.

Define an action of B on $\mathbb{P}^2$ by

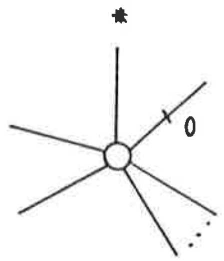
$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} (z_0:z_1:z_2) = (\alpha z_0+\beta z_1:\alpha^{-1}z_1:z_2).$$

Then $\mathbb{P}^2$ has 5 orbits: one open orbit isomorphic to B, 2 orbits of dimension one, and 2 fixed points. We pick a base point in the open orbit, for example (0:1:1). This defines an embedding of B. We will show that the diagram of the embedding is

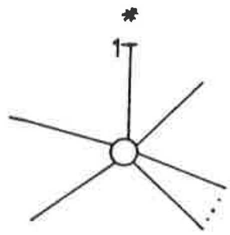


To see this, first we consider the orbits of dimension one: $C_1=\{(a:0:1) \mid a\in k^*\}$ and $C_2=\{(a:1:0) \mid a\in k\}$. Each orbit has a valuation ring, and it is obtained from a formally divergent curve $\lambda(t)$ [9].

Let $\lambda_1(t)= \begin{pmatrix} t^{-1} & 1 \\ 0 & t \end{pmatrix}$. By our choice of the base point, the image under the injective morphism $B \hookrightarrow \mathbb{P}^2$ is (0:1:1). So $\lambda_1(t) \longrightarrow \begin{pmatrix} t^{-1} & 1 \\ 0 & t \end{pmatrix}(0:1:1)$ = (1:t:1). Now $\lim_{t\to 0} \lambda_1(t)=(1:0:1)$ which belongs to C_1. So the valuation ring of $\overline{C}_1$ in $\mathbb{P}^2$ is given by v_1 with $v_1(\alpha)=-1$ and $v_1(\beta)=0$. We extend v_1 to a valuation in $\mathcal{V}(G)$. Now $G= SL(2,k)$ with coordinates a_{ij}, $i=1,2$, $j=1,2$. Thus $v_1(a_{21})=v_1(a_{11})=-1$, and $v_1(a_{22})=v_1(a_{12})=0$; so $v_1=v(D,0)$, where $f_D=a_{22}$, so $D\neq B$. In other words, the valuation ring of C_1 in $\mathbb{P}^2$ is represented by



Now let $\lambda_2(t)=\begin{pmatrix} t & t^{-1} \\ 0 & t^{-1} \end{pmatrix}$. Then $\lambda_2(t)\longrightarrow \begin{pmatrix} t & t^{-1} \\ 0 & t^{-1} \end{pmatrix}(0:1:1)=(1:1:t)$. So $\lim_{t\to 0} \lambda_2(t)=(1:1:0)$, which belongs to C_2. Thus the valuation ring of C_2 in $\mathbb{P}^2$ is v_2 with $v_2(\alpha)= 1$ and $v_2(\beta)= -1$. If we extend v_2 to a valuation in $\mathcal{V}(G)$, we find $v_2= v(B,1)$; so the valuation ring of $\overline{C}_2$ in $\mathbb{P}^2$ is represented by



Now let us look at the two fixed points: $P_1=(0:0:1)$ and $P_2=(1:0:0)$. Set $\mathcal{L}_i$ equal to the locality of P_i, i=1,2; we see that $\mathcal{V}_{\mathcal{L}_1}=\{v_1\}$, since P_1 is in the closure of C_1, and $\mathcal{V}_{\mathcal{L}_2}=\{v_1,v_2\}$, since P_2 is in the closures of both C_1 and C_2. So $\mathcal{L}_1$ must be a locality of type B_+, and $\mathcal{L}_2$ must be of type A_2. The diagram of this embedding is indeed the one shown earlier.

§4. Morphisms of embeddings of G/Γ and B/Γ

If X is a normal embedding of G/Γ (resp. B/Γ), we define $L(X) \subset \mathcal{L}_1^n(G/\Gamma)$ (resp. $\mathcal{L}_1^n(B/\Gamma)$) to be the set of localities of orbits of X.

Proposition 4.4.1. Let $\Gamma \subset G$ be a finite subgroup, and let X and X′ be two normal embeddings of G/Γ. Then the identity map on G/Γ extends to a (necessarily unique) G-morphism $\varphi: X \longrightarrow X'$ if and only if for each $\ell \in L(X)$ there exists $\ell' \in L(X')$ such that $F_\ell \subset F_{\ell'}$ and if ℓ is of type B_0 then so is ℓ', and if ℓ is of type B_+ (resp. B_-) then ℓ' is either of the same type or of type B_0. If φ exists, then it is proper if and only if $\bigsqcup_{\ell \in L(X)} F_\ell = \bigsqcup_{\ell \in L(X')} F_\ell$.

Proof.

We use the notation from Chapter I.

It is clear that if φ exists, it is unique. Also the statement about the properness is proven in [9], section 6.4.

Now suppose the condition given in the proposition is verified. Then it is easily checked that ${}^B\mathcal{D}_\ell \subset {}^B\mathcal{D}_{\ell'}$. This means we can choose $\mathcal{D}, \mathcal{D}' \subset \mathbb{P}^1/\Gamma$ such that $\mathcal{O}_\ell$ (resp. $\mathcal{O}_{\ell'}$) is a local ring of $A(\mathcal{D},\mathcal{V}_\ell)$ (resp. $A(\mathcal{D}',\mathcal{V}_{\ell'})$) and $\mathcal{D} \subset \mathcal{D}'$. Therefore $A(\mathcal{D}) \supset A(\mathcal{D}')$. Now if $v \in \mathcal{V}_\ell$, then either $v \in \mathcal{V}_{\ell'}$ or $\mathcal{O}_v$ dominates $\mathcal{O}_\ell$. This implies that we have $A(\mathcal{D},\mathcal{V}_\ell) \supset A(\mathcal{D}',\mathcal{V}_{\ell'})$. This inclusion induces a morphism $\operatorname{Spec} A(\mathcal{D},\mathcal{V}_\ell) \longrightarrow \operatorname{Spec} A(\mathcal{D}',\mathcal{V}_{\ell'})$, which is compatible with the identity map on G/Γ. We extend this morphism to $G\cdot\operatorname{Spec} A(\mathcal{D},\mathcal{V}_\ell) \longrightarrow G\cdot\operatorname{Spec} A(\mathcal{D}',\mathcal{V}_{\ell'})$. For all $\ell \in L(X)$, these morphisms are compatible, so they define a morphism $\varphi: X \longrightarrow X'$.

Conversely, if φ exists, let Y be an orbit of X with locality ℓ. Then $\mathcal{O}_\ell$ dominates $\mathcal{O}_{\ell'}$, where ℓ' is the locality of $\varphi(Y)$. This implies first of all that $F_\ell \subset F_{\ell'}$. Also if $D \in {}^B\mathcal{D}_\ell$, then $\overline{D} \supset Y$; thus $\overline{D} = \overline{\varphi(D)} \supset \varphi(Y)$, which means that $D \in {}^B\mathcal{D}_{\ell'}$. Thus we must have the requirement given in the proposition.

□

<u>Corollary 4.4.2</u>. Let $\Gamma \subset B$ be a finite subgroup, and let X and X′ be two normal embeddings of B/Γ. Then the identity map on B/Γ extends to a (necessarily unique) B-morphism $\varphi: X \longrightarrow X'$ if and only if for each $\ell \in L(X)$ there exists $\ell' \in L(X')$ such that $F_\ell \subset F_{\ell'}$. If φ exists, then it is proper if and only if

$$\bigsqcup_{\ell \in L(X)} F_\ell = \bigsqcup_{\ell \in L(X')} F_\ell .$$

<u>Proof</u>.

This morphism φ exists if and only if there is a morphism $G *_B X \longrightarrow G *_B X'$ which extends the identity map of G/Γ.

□

(This corollary can also be proven directly by showing that $F_\ell \subset F_{\ell'}$ if and only if $\mathcal{O}_\ell$ dominates $\mathcal{O}_{\ell'}$.)

Given a normal B/Γ-embedding X, we define the skeleton of X , denoted Sk(X), to be the set of valuations $v \in \mathcal{V}(B/\Gamma)$ such that v is that locality of an irreducible subvariety of dimension one in X.

Now suppose X is a smooth B/Γ-embedding with fixed point P. Denote by $\tilde{X}$ the variety obtained by blowing up P in X. Then $\tilde{X}$ is smooth, and the action of B on X induces an action on $\tilde{X}$ giving $\tilde{X}$ the structure of a B/Γ-embedding. We want to describe the diagram of $\tilde{X}$ given that of X. We know that:

(i) $\bigsqcup_{\ell \in L(X)} F_\ell = \bigsqcup_{\ell \in L(\tilde{X})} F_\ell$ since $\tilde{X} \longrightarrow X$ is proper;

(ii) if v is the valuation of the exceptional divisor, then $v \in \mathcal{V}(B/\Gamma)$ and v dominates P in X;

(iii) $Sk(\tilde{X}) = Sk(X) \cup \{v\}$ (which with (i) implies that for each $\ell \in L(X)$ there is an $\ell' \in L(\tilde{X})$ whose facette is contained in the facette of ℓ);

and (iv) $L(\tilde{X}) \subset S(B/\Gamma)$ since $\tilde{X}$ is smooth.

By the following corollary, these four properties destermine the diagram of $\tilde{X}$.

<u>Corollary 4.4.3</u>. Suppose X is a smooth B/Γ-embedding with fixed point P; then the blow up of X at P is the only smooth embedding $\tilde{X}$ such that

$\bigsqcup_{\ell\in L(X)} \mathcal{F}_\ell = \bigsqcup_{\ell\in L(\tilde{X})} \mathcal{F}_\ell$ and $Sk(\tilde{X})=Sk(X)\cup\{v\}$ for some $v\in\mathcal{V}(B/\Gamma)$ which dominates P in X.

<u>Proof</u>

Denote the blow up of P in X by $\tilde{X}$; by the remarks above it satisfies the given requirements. Now suppose $\tilde{X}_0$ is another such embedding. By Corollary 4.4.2 we have

$$\begin{array}{ccc} & & \tilde{X} \\ & & \downarrow \\ \tilde{X}_0 & \xrightarrow{\varphi} & X \end{array}$$

and $\varphi^{-1}(P)$ is a curve in $\tilde{X}_0$. By the universal property of blowing up for smooth surfaces, there exists $\tilde{\varphi}\colon \tilde{X}_0 \longrightarrow \tilde{X}$ such that φ factors through $\tilde{\varphi}$. By Corollary 4.4.2 again, $\tilde{\varphi}$ must be an isomorphism.

□

<u>§5. Blowing up smooth embeddings of B and B/{±e}</u>

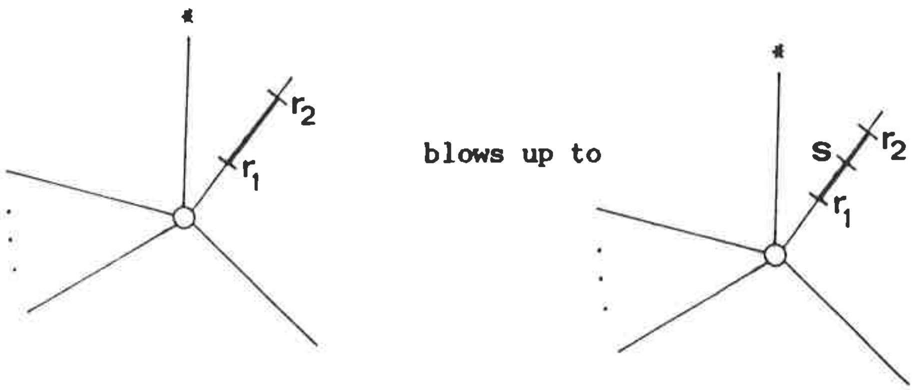
In this section, we will give explicitly the diagrams obtained by blowing up the different types of orbits for $\Gamma=\{e\}$ and $\Gamma=\{\pm e\}$.

<u>B-embeddings</u>:

Let X be a smooth B-embedding with fixed point P. From section 4 together with the classification of smooth localities in section 2.2, we can give explicitly the diagram for $\tilde{X}$ knowing the diagram for X.. For example, suppose the locality ℓ of P is of type AB with $\mathcal{V}_\ell=\{v(D,r_1),v(D,r_2)\}$ where $r_i=\dfrac{p_i}{q_i}$, i=1,2; then since the locality is smooth, we know that $|r_2-r_1|=(q_1q_2)^{-1}$ (in other words, r_1 and r_2 are neighbors in a Farey sequence). The locality of the exceptional divisor must belong to $\mathcal{F}_\ell$. So $L(\tilde{X}) = (L(X)-\{\ell\})\cup\{v,\ell_1,\ell_2\}$, where $v\in\mathcal{F}_\ell$, $\ell_1,\ell_2\in S(B)$ each of type AB. If

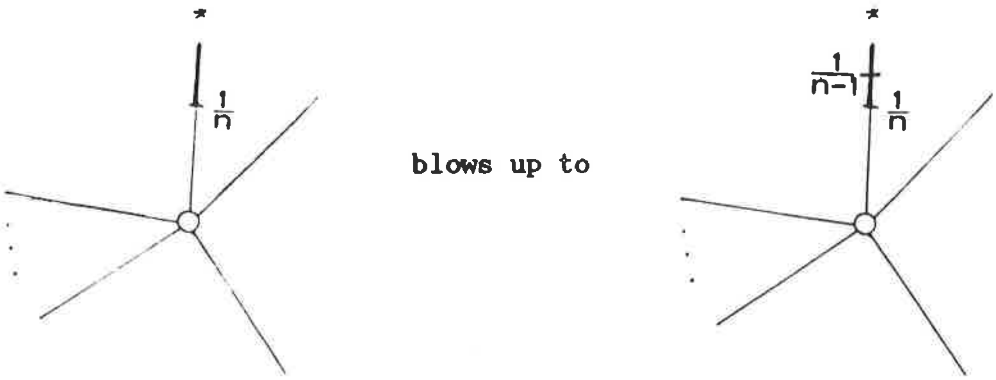
$v=v(D,s)$, then $\mathcal{V}_{\ell_1} = \{v(D,r_1), v(D,s)\}$ and $\mathcal{V}_{\ell_2} = \{v(D,s), v(D,r_2)\}$. The only way ℓ_1 and ℓ_2 can be smooth is if $s = \frac{p_1+p_2}{q_1+q_2}$. Similarly, if the locality is another type, the diagram of $\widetilde{X}$ is completely determined. We give now a complete list of how to blow up each type of smooth fixed point.

Type AB.

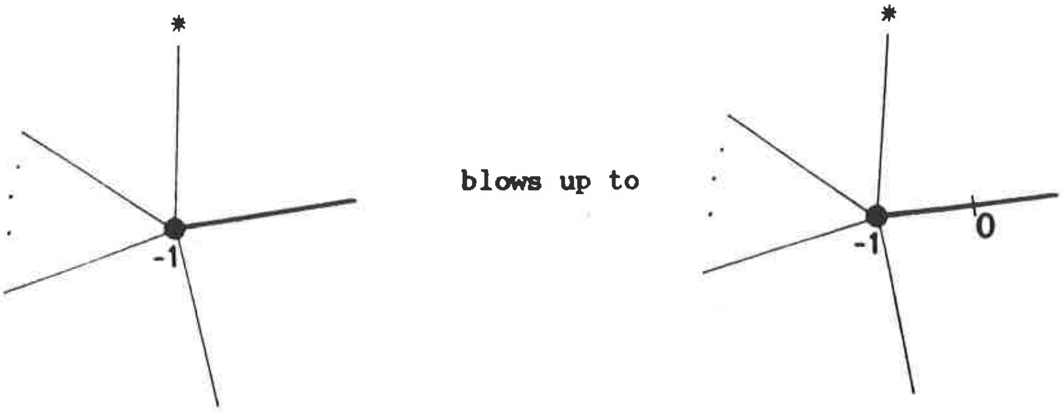


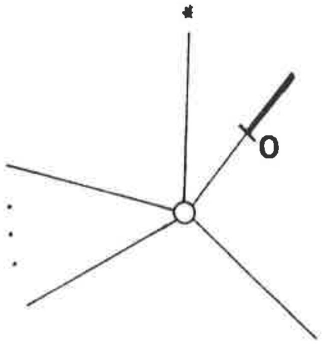
where $r_1=\frac{p_1}{q_1}$, $r_2=\frac{p_2}{q_2}$ and $s=\frac{p_1+p_2}{q_1+q_2}$.

Type B_-.

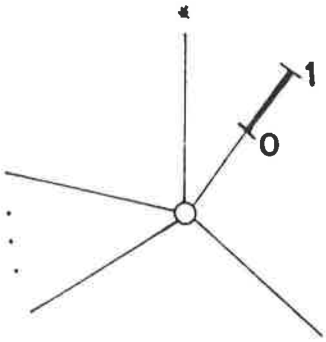


Type B_+.

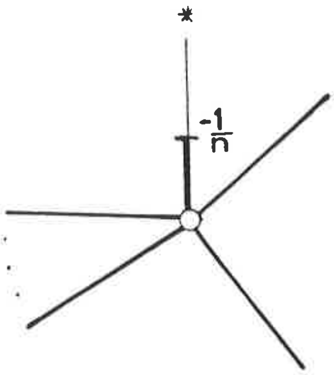




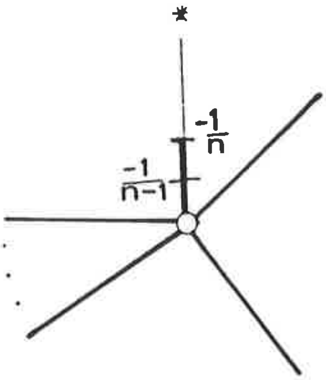
blows up to



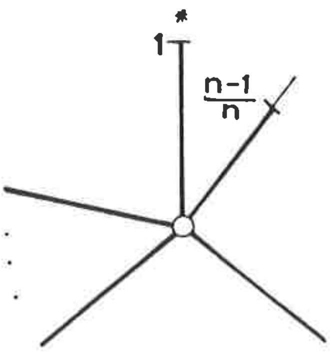
Type A_1.



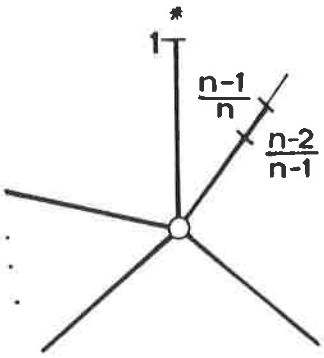
blows up to

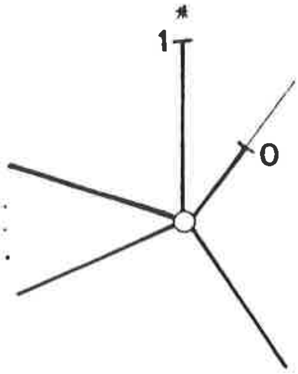


Type A_2.

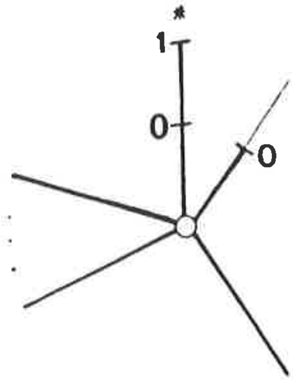


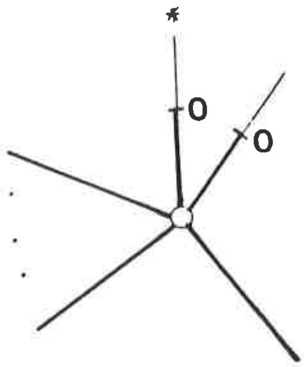
blows up to



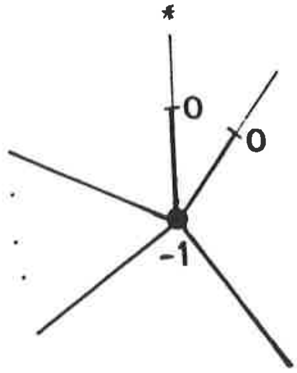


blows up to



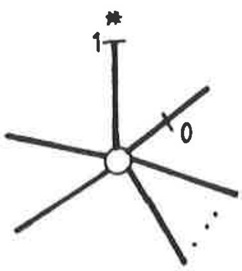


blows up to



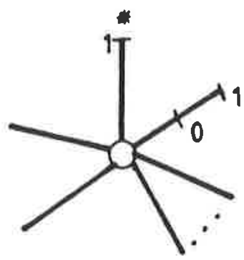
This gives us important information about the geometric structure of B-embeddings.

Example We start with the example given in section 3:

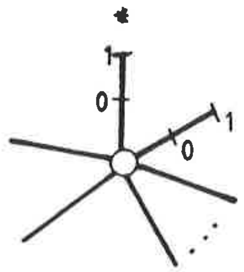


represents an embedding isomorphic to $\mathbb{P}^2$.

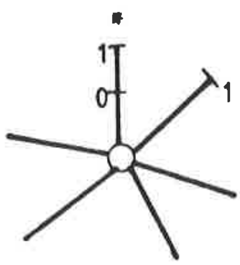
Now we know if we start with $\mathbb{P}^2$ and blow up two distinct points and then blow down the strict transform of the line joining those two points, we obtain a copy of $\mathbb{P}^1 \times \mathbb{P}^1$. Let us do this with our embedding isomorphic to $\mathbb{P}^2$. We will blow up the two fixed points and then blow down the line connecting them. First we blow up the type B_+ orbit. The new diagram is given by



Now we blow up the type A_2 orbit. Now the diagram is



Lastly we blow down the strict transform of the line connecting the original two fixed points. The diagram is

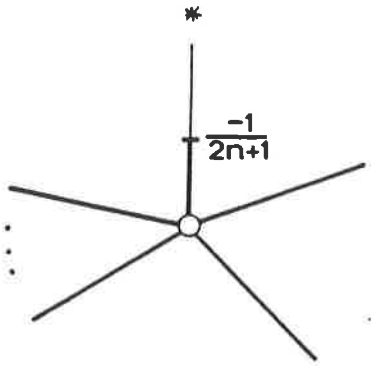


so the embedding with this diagram is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

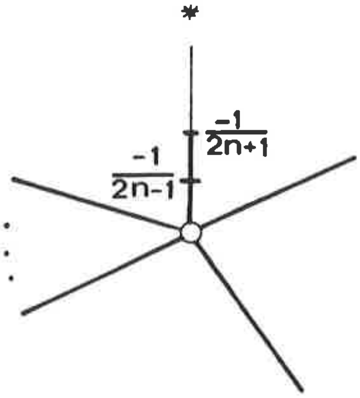
<u>B/{±e}-embeddings</u>:

We can do the same thing for the case $\Gamma=\{\pm e\}$ as we did for $\Gamma=\{e\}$. We must check how to blow up each type of fixed point given in Proposition 2.3.1. The result follows.

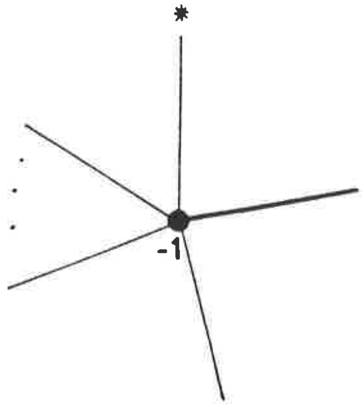
Type A_1.



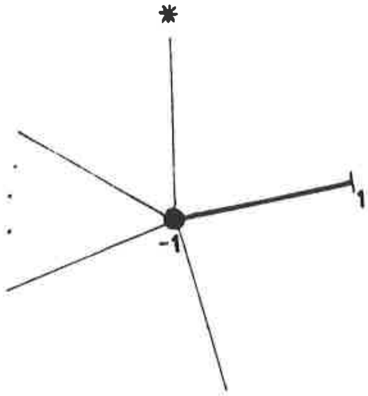
blows up to



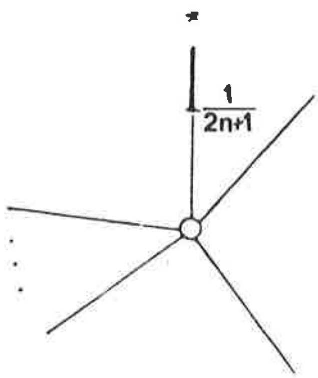
Type B_+.



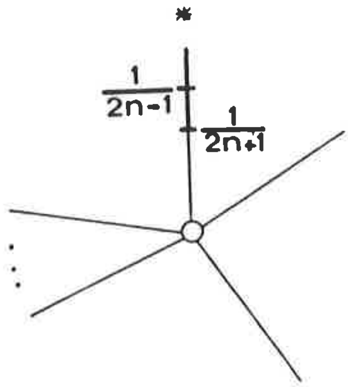
blows up to



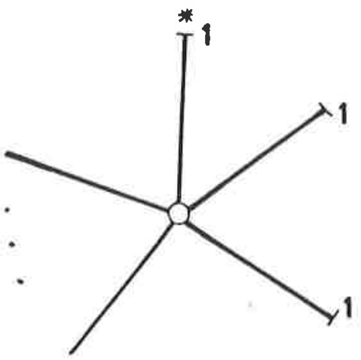
Type B_-.



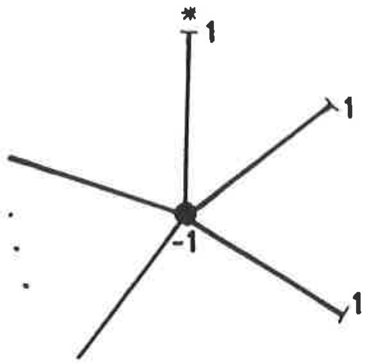
blows up to



Type A_3.

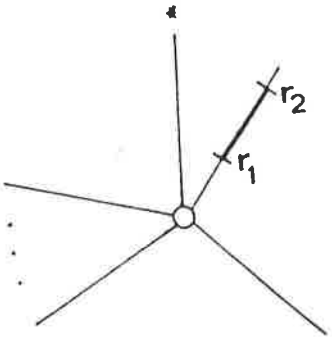


blows up to

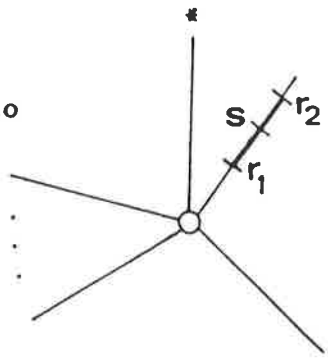


Type AB.

For this case, write $r_i = \frac{p_i}{q_i}$ in lowest form such that 2 divides $q_i - p_i$ and $q_i > 0$, $i=1,2$. Then



blows up to



where

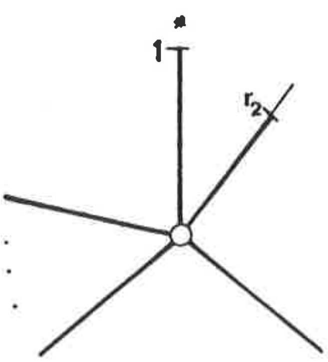
$r_1 = \frac{p_1}{q_1}$,

$r_2 = \frac{p_2}{q_2}$ and

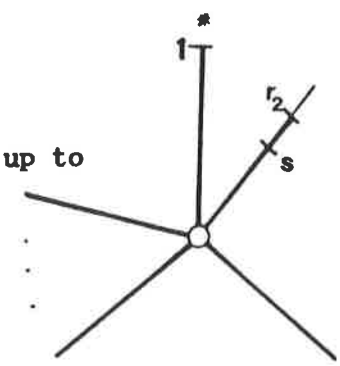
$s = \frac{p_1+p_2}{q_1+q_2}$.

Type A_2.

Write r_i as in type AB. Then one finds that the only smooth embeddings of this type are those with $r_1=1$ and $q_2-p_2=2$ or vice versa.



blows up to



where

$r_2 = \frac{p_2}{q_2}$ and

$s = \frac{p_2-1}{q_2-1}$

(Note that if $r_2 = 0 = \frac{0}{2}$, then $s=-1$.)

§6. The minimal B/Γ-embeddings

In this section we will give the diagrams for the minimal embeddings found in Chapter III. There are two ways to do this. First one can calculate the valuations of the one-dimensional stable subvarieties of each embedding found in Chapter III. Then since the union of the facettes of all the orbits must cover the whole diagram (and since there is no confusion between the type B_+ and B_- orbits), one knows the diagram of the embedding. The other method is to use the fact that any smooth B/Γ-embedding can be obtained by blowing up and down stable subvarieties of any other smooth B/Γ-embedding. (This is proven in the same way as showing that a birational equivalence of smooth surface is a composition of blow ups and blow downs.) So one can start with one embedding whose diagram is known (such as the one given in section 3 into $\mathbb{P}^2$) and by blowing up and down, we obtain all the smooth embeddings. One can keep track of which are minimal models (as in the example in section 4 where we find the diagram of an embedding into $\mathbb{P}^1 \times \mathbb{P}^1$.

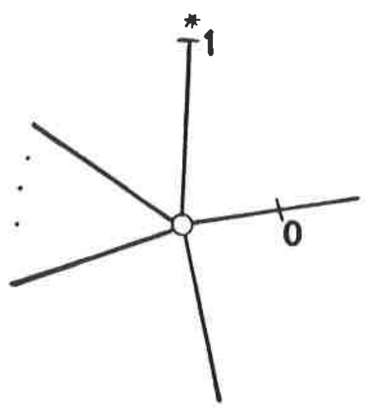
The minimal embeddings of B

In Chapter III we found there are two embeddings (up to equivalence) in $\mathbb{P}^2$, one in $\mathbb{P}^1 \times \mathbb{P}^1$, and n+3 in $\mathbb{F}_n$ $n \geq 1$. (We check $\mathbb{F}_1$ even though it is not a minimal model to make sure the result is consistent.) For each of these embeddings we give the diagram.. For the actions of B on the minimal surfaces we use the notation of Chapter III.

$\mathbb{P}^2$:

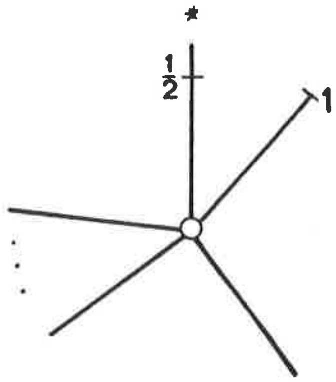
$$B \longrightarrow PGL(3)$$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha^{-1} \end{bmatrix} :$$



(The brackets indicate the class in PGL(3).)

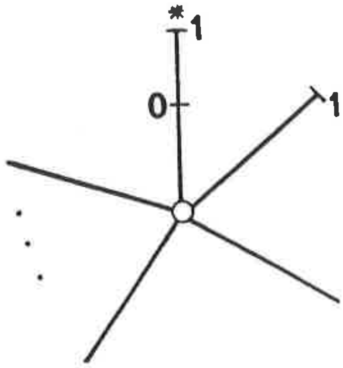
$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha^{-1} \end{bmatrix} :$$



$\mathbb{P}^1 \times \mathbb{P}^1$:

$$B \longrightarrow PGL(2)\times PGL(2) \subset Aut(\mathbb{P}^1 \times \mathbb{P}^1)$$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \right):$$

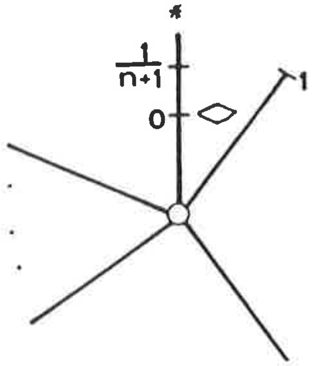


$\mathbb{F}_n$, $n \geq 1$:

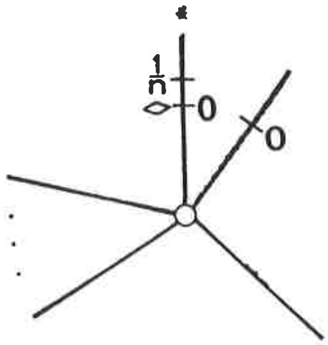
In Chapter III, we actually described the embeddings of B in X_n, the variety obtained by blowing down the irreducible curve E_n of negative self-intersection. So here we give (i) the B-module isomorphic to k^{n+2} which induces a B-action on $X_n \subset \mathbb{P}^{n+1}$ as given in Chapter III, and (ii) the embedding in $\mathbb{F}_n$ obtained by blowing up the "center" of X_n. We mark the valuation of E_n with a small diamond.

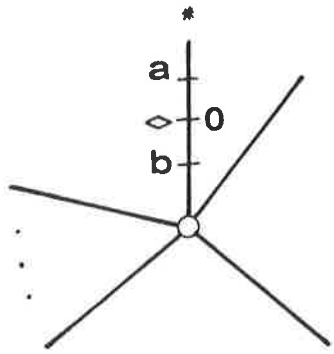
<u>**Case 1**</u>

$$k^2 \oplus (k,\alpha^{p+1}) \oplus \bigoplus_{\substack{j=0 \\ j\neq p}}^{n} (k,\alpha^j) \ :$$



if p=n;





where

$a = \frac{1}{p+1}$,

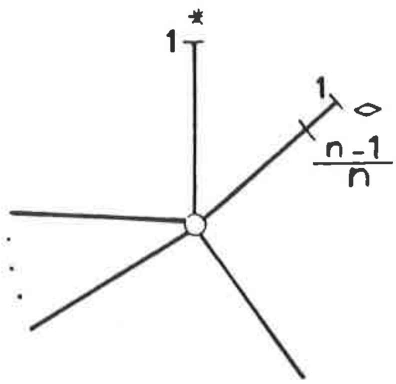
$b = \frac{-1}{n-p-1}$

if p= n-1;

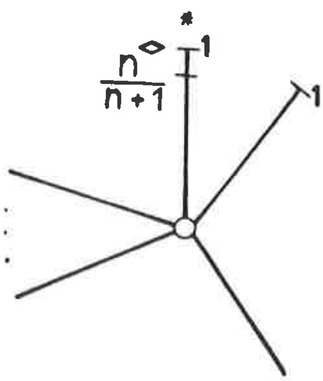
if p<n-1.

<u>**Case 2.**</u>

$(k,\alpha^{-n+1}) \oplus S^n(k^2)$:



$(k,\alpha^{-n-1}) \oplus S^n(k^2)$:



Note that for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ we always have two fixed points; for Case 1 of $\mathbb{F}_n$ we have three fixed points if $p=n$ or $p = 0 \neq n-2$, a curve of fixed points (a fibre of $\mathbb{F}_n \longrightarrow \mathbb{P}^1$) if $p = n-2$, and otherwise four fixed points, and for Case 2 we have two fixed points.

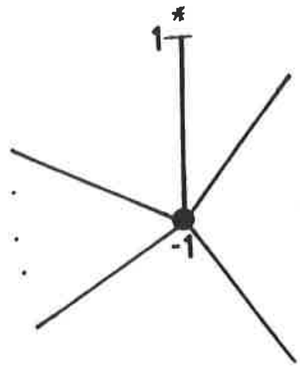
<u>The minimal embeddings of $B/\{\pm e\}$</u>

Now there are two embeddings in $\mathbb{P}^2$, two in $\mathbb{P}^1 \times \mathbb{P}^1$, and n+3 in $\mathbb{F}_n$ $n \geq 1$. We give the diagrams for these embeddings.

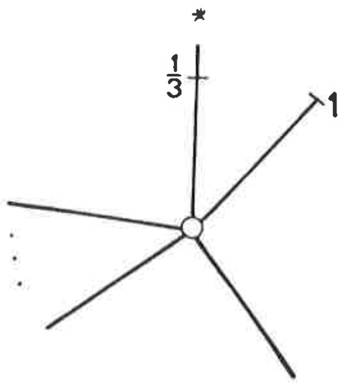
$\mathbb{P}^2$:

$$B \longrightarrow PGL(3)$$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha^{-1} \end{bmatrix} :$$



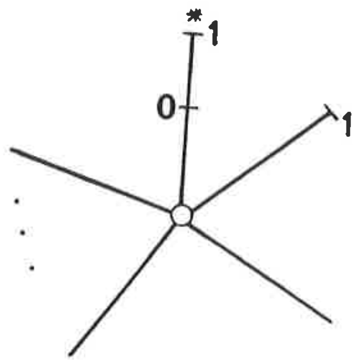
$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha^{-1} \end{bmatrix} :$$



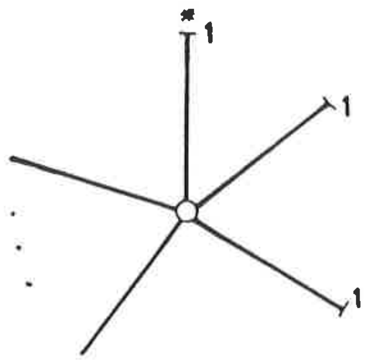
$\mathbb{P}^1 \times \mathbb{P}^1$:

$$B \longrightarrow PGL(2) \times PGL(2) \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \left(\begin{bmatrix} \alpha^2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \right):$$



$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \longmapsto \left(\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \right):$$

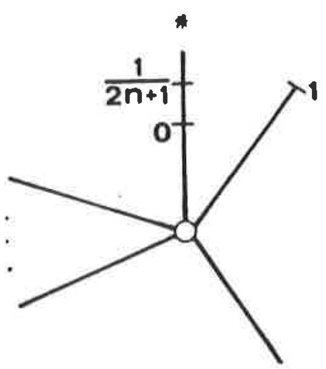


(Here the three stable curves intesect in the unique fixed point.)

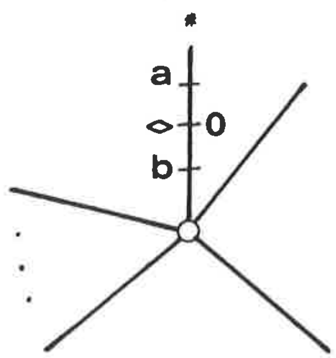
$\mathbb{F}_n$, $n \geq 1$:

<u>Case 1</u>

$$k^2 \oplus (k, \alpha^{2p+1}) \oplus \bigoplus_{\substack{j=0 \\ j \neq p}}^{n} (k, \alpha^{2j}) :$$



if p=n;



if p<n.

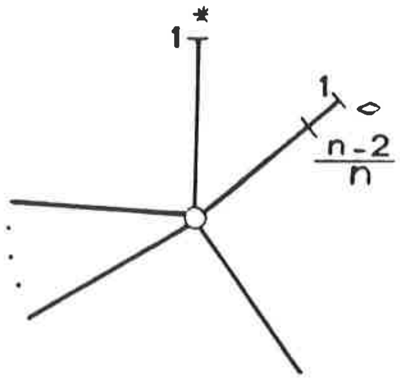
where

$a = \frac{1}{2p+1}$,

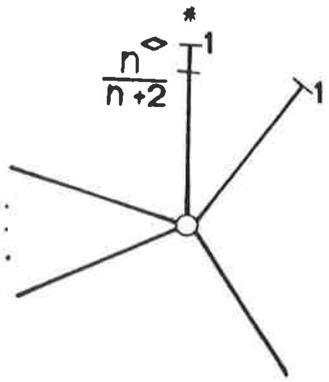
$b = \frac{1}{2(n-p)-1}$

Case 2.

$(k,\alpha^{-n+2}) \oplus S^n(k^2)$:



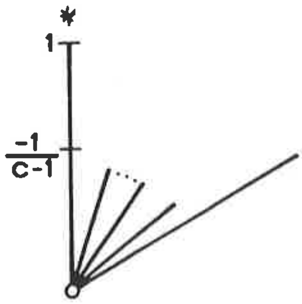
$(k,\alpha^{-n-2}) \oplus S^n(k^2)$:

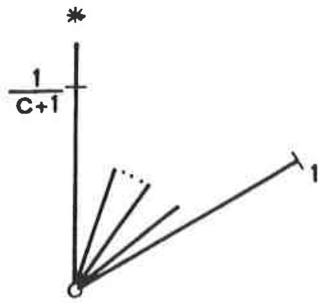


Minimal embeddings of B/Γ , $|\Gamma|\geq 3$.

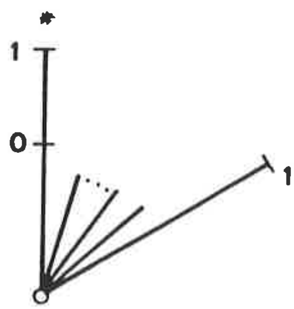
Let $|\Gamma| = c$. Here we give just the diagrams.

$\mathbb{P}^2$:

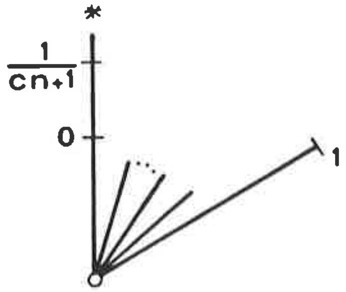


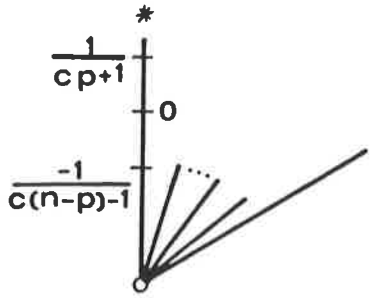


$\mathbb{P}^1 \times \mathbb{P}^1$:

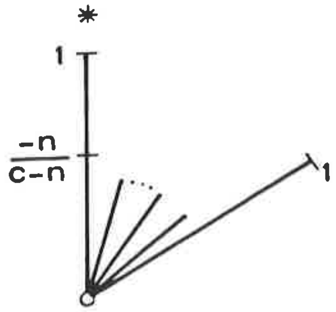


$\mathbb{F}_n$, $n \geq 1$:

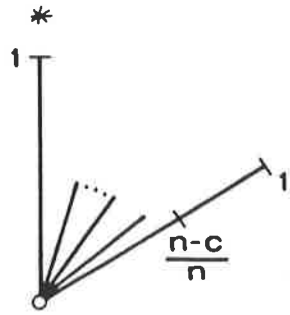




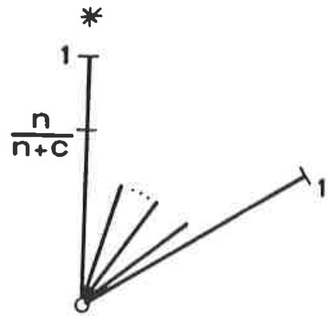
$p=0,\ldots,n-1$



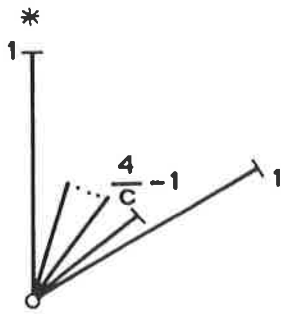
if $c \geq 2n$



if $c < 2n$



and if c is even, then there is one more embedding for $\mathbb{F}_{c/2-1}$:



§7. Minimal embeddings of SL(2,k) and PGL(2,k)

Let X′ be a smooth embedding of G or G/{±e} with orbit Y. Then there is a G-stable open subvariety X′′ containing Y such that X′′ is of the form $G*_B X$

for some Borel subgroup of G , and X is a smooth embedding of B or B/{±e}. (One must simply choose B such that Y is not in the closure of B in X'.) We can use this information to study local properties of X'.

For example if Y is of dimension one, we can blow up Y just as we blow up Y∩X in X. This allows us to find the diagrams of minimal smooth G and G/{±e}. We use the following proposition.

<u>Proposition 4.7.1</u>. Let Γ be the group {e} or {±e}. Suppose X_1 and X_2 are smooth G/Γ-embeddings with a G-morphism $\varphi: X_1 \longrightarrow X_2$ extending the identity map on G/Γ. Suppose also that $Sk(X_1) \supset Sk(X_2) \cup \{v\}$ where v dominates a closed orbit Y of X_2. Then there exists a G-morphism $\tilde{\varphi}: X_1 \longrightarrow \tilde{X}_2$, where $\tilde{X}_2$ is the blow up of X_2 at Y, such that φ factors through $\tilde{\varphi}$.

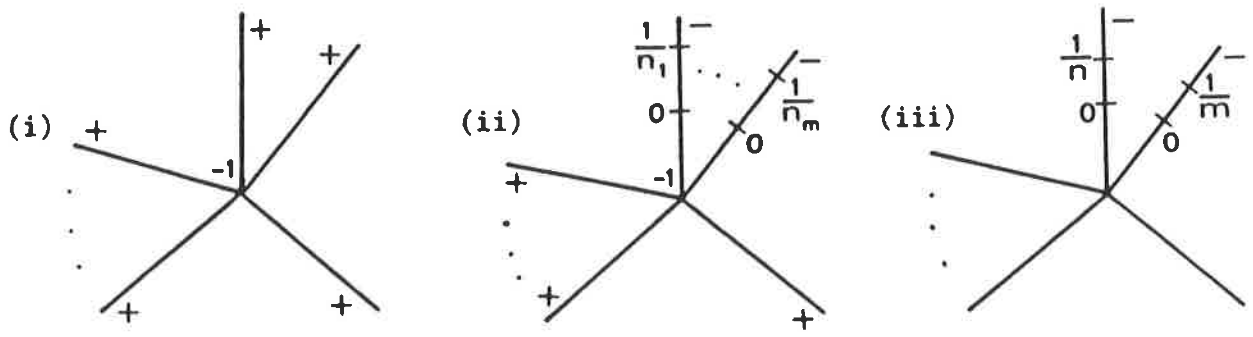
<u>Proof.</u>

Since blowing up is a local property, we can assume that X_2 is of the form $G*_B X$. Here the proposition is true by the universal property of blowing up on smooth surfaces.

□

We find

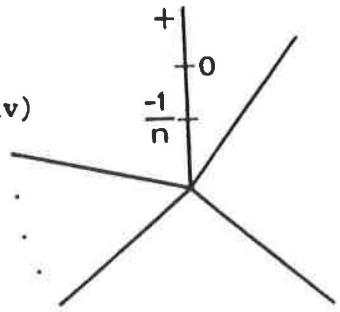
<u>Minimal smooth SL(2,k)-embeddings</u>



$n_i \geq 2$, i=1,...,m, $m \geq 3$

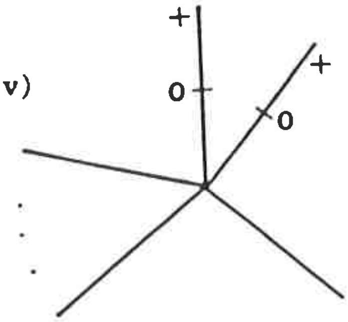
n,m≥2

(iv)

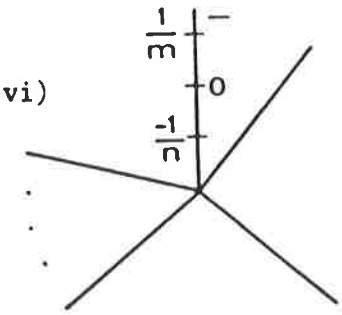


$n \geq 2$

(v)



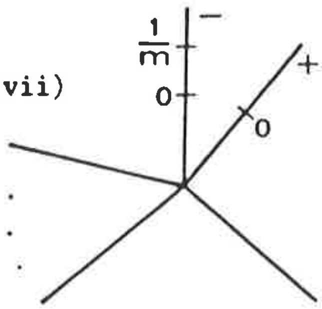
(vi)



$n \geq 1, m \geq 2$

$(G *_B \mathbb{F}_{n+m})$

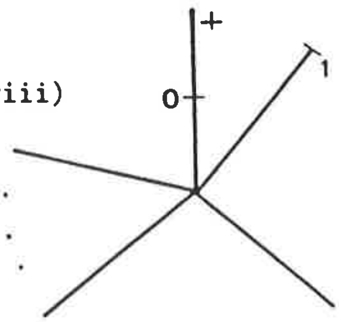
(vii)



$m \geq 2$

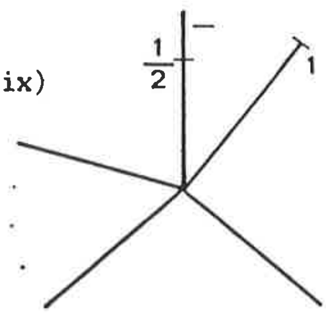
$(G *_B \mathbb{F}_m)$

(viii)



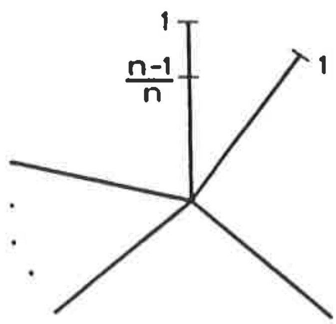
$(G *_B \mathbb{P}^2)$

(ix)



$(G *_B \mathbb{P}^2)$

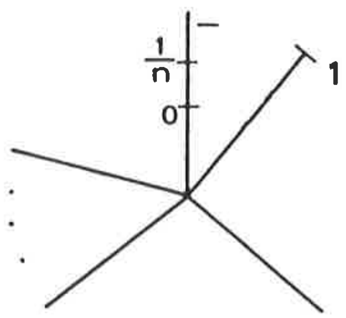
(x)



$n \geq 3$,

$(G *_B \mathbb{F}_n$ or $G *_B \mathbb{F}_{n-1}$ depending on the choice of the Borel subgroup.)

(xi)



$n \geq 3$.

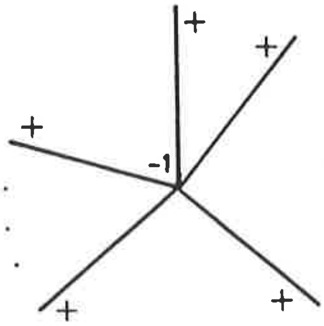
$(G *_B \mathbb{F}_{n-1})$

The only ones of these which are not of the form $G *_B X$ for some Borel subgroup B are (i) - (v). The first one is obtained by

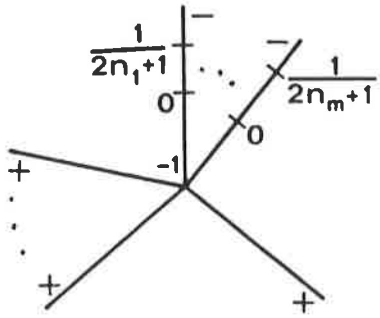
G $\hookrightarrow$ M(2,k) $\hookrightarrow$ $\mathbb{P}^4$, where M(2,k) is the vector space of 2 × 2 matrices over k. The action of G on $\mathbb{P}^4$ is induced by left multiplication, and the embedding is the closure of the image of G in $\mathbb{P}^4$. So this is a projective embedding.

Minimal smooth PGL(2,k)-embeddings

This time I only draw those that are not of the form $G*_B X$. There are only two types:



and



$n_i \geq 2$, $i=1,\ldots,m$, $m \geq 2$.

The first is obtained by the action of G on $\mathbb{P}(M(2,k)) \cong \mathbb{P}^3$ (induced by left multiplication). So this embedding is projective. The second one can show is always nonprojective. An example of such an embedding is given in [8].

Note also that for any smooth (P)SL(2,k)-embedding X′ ,we can blow up X′ a finite number of times such that it is of the form $G*_B X$.

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